Semicontinuity properties of Kazhdan-Lusztig cells

Cédric Bonnafé

SEMICONTINUITY PROPERTIES OF KAZHDAN-LUSZTIG CELLS

CÉDRIC BONNAFÉ

Abstract. Computations in small Coxeter groups or dihedral groups suggest that the partition into Kazhdan-Lusztig cells with unequal parameters should obey to some semicontinuity phenomenon (as the parameters vary). The aim of this paper is to provide a rigorous theoretical background for supporting this intuition that will allow to state several precise conjectures.

Let (W, S) be a Coxeter group and assume, for simplification in this introduction, that S is finite and that $S = S_1 \dot{\cup} S_2$, where S_1 and S_2 are two non-empty subsets of S such that no element of S_1 is conjugate to an element of S_2. Let $\ell : W \to \mathbb{N} = \{0, 1, 2, \dots\}$ be the length function and $\ell_i : W \to \mathbb{N}$ the S_i-length function (if $w \in W$, $\ell_i(w)$ is the number of occurrences of elements of S_i in a reduced decomposition of w). Let us fix two non-zero natural numbers a and b and let $L_{a,b} : W \to \mathbb{Z}$, $w \mapsto a\ell_1(w) + b\ell_2(w)$. Then $L_{a,b}$ is a *weight function* (in the sense of Lusztig) so it is possible to define a partition of W into left Kazhdan-Lusztig cells. It is clear that this partition depends only on b/a: we shall denote it by $\mathscr{L}_{b/a}(W)$. Explicit computations suggest the following conjecture:

Conjecture 0. *There exist rational numbers $0 < r_1 < \cdots < r_m$ (depending only on (W, S)) such that (setting $r_0 = 0$ and $r_{m+1} = +\infty$), if θ and θ' are two positive rational numbers, then:*

(a) *If $0 \leqslant i \leqslant m$ and $r_i < \theta, \theta' < r_{i+1}$, then $\mathscr{L}_\theta(W) = \mathscr{L}_{\theta'}(W)$.*

(b) *If $1 \leqslant i \leqslant m$ and $r_{i-1} < \theta < r_i < \theta' < r_{i+1}$, then $\mathscr{L}_{r_i}(W)$ is the finest partition of W which is less fine than $\mathscr{L}_\theta(W)$ and less fine than $\mathscr{L}_{\theta'}(W)$.*

Remarks - (1) One can obviously state similar conjectures for the partitions into right or two-sided cells.

(2) In the case where W is finite, the existence of rational numbers $0 < r_1 < \cdots < r_m$ satisfying (a) is easy (see Proposition 5.7). However, even in this case, the statement (b) is still a conjecture. □

Date: March 24, 2010.

1991 *Mathematics Subject Classification.* According to the 2000 classification: Primary 20C08; Secondary 20C15.

The author is partly supported by the ANR (Project No JC07-192339).

The aim of this paper is to provide a theoretical background that allows us to generalize this conjecture in the following two directions:

- It is of course possible to work with a partition of S into more than two subsets (this does not occur if W is finite and irreducible).
- One can also be interested in weight functions with values in any totally ordered abelian group, and whose values on simple reflections are not necessarily positive.

These two generalisations lead us to define equivalence classes of weight functions (in the easy case detailed in this introduction, this has been done by working with the ratio b/a instead of the pair (a, b)) and to use a topology on the set of such equivalence classes. This is done by using the notion of *positive subsets* of an abelian group (as defined in [4, §1]): we shall review some of the results of [4] in an Appendix.

This paper is organized as follows. After two sections devoted to recollections of well-known facts about Hecke algebras and Kazhdan-Lusztig theory with unequal parameters, we shall state our main conjecture in the third one: it is only concerned with the partition into cells and will be given into different forms (see Conjectures A, A' and A"). In the fourth section, we shall also state a conjecture saying, roughly speaking, that Conjecture A is compatible with the construction of *cell representations* (see Conjecture B). In the last section, we illustrate these conjectures by giving a detailed account of the following examples:

EXAMPLES - (1) *Dihedral groups:* If $|S| = 2$ and if $|S_1| = |S_2| = 1$, then the Conjecture 0 holds by taking $m = 1$ and $r_1 = 1$ (see [18, §8.8]).

(2) *Type F_4:* If (W, S) is of type F_4 and if $|S_1| = |S_2| = 2$, then the Conjecture 0 holds by taking $m = 3$ and $r_1 = 1/2$, $r_2 = 1$ et $r_3 = 2$ (see [11, Corollaire 4.8]).

(3) *Type B_n:* Assume that (W, S) is of type B_n (with $n \geqslant 2$) and that $|S_1| = n-1$ and $|S_2| = 1$. In [5, Conjectures A and B], the Conjecture 0 is made more precise: it should be sufficient to take $m = n - 1$ and $r_i = i$. This has been checked for $n \leqslant 6$.

(4) *Type $\tilde{G}_2$:* Assume that (W, S) is the affine Weyl group of type $\tilde{G}_2$ and that S_1 and S_2 are chosen in such a way that $|S_1| = 2$ and $|S_2| = 1$. Then Guilhot [15] has shown that the Conjecture 0 holds by taking $m = 3$ and $r_1/1$, $r_2 = 3/2$ and $r_3 = 2$.

(5) *Type $\tilde{B}_2$:* Guilhot [15] has also shown that Conjecture 0 holds if (W, S) is an affine Weyl group of type $\tilde{B}_2$. In this case, there are several possibilities for the choice of the partition $S = S_1 \dot{\cup} S_2$, but Guilhot has proved that Conjecture 0 holds for all possible choices. □

Acknowledgements. Part of this work was done while the author stayed at the MSRI during the winter 2008. The author wishes to thank the Institute for its hospitality and the organizers of the two programs for their invitation. The author also thanks M. Geck, L. Iancu and J. Guilhot for many fruitful discussions.

CONTENTS

1. HECKE ALGEBRAS

1.A. **Preliminaries.** Let (W,S) be a Coxeter system (S and W can be infinite). If s, $t \in S$, we shall write $s \sim t$ if they are conjugate in W. Let $\bar{S} = S/\sim$. If $s \in S$, we denote by $\bar{s}$ its class in $\bar{S}$. Let $\mathbb{Z}[\bar{S}]$ the free $\mathbb{Z}$-module with basis $\bar{S}$.

Let $\ell : W \to \mathbb{N}$ be the length function associated with S. If $\omega \in \bar{S}$ and if $w \in W$, we denote by $\ell_\omega(w)$ the number of occurrences of elements of ω in a reduced expression of w (it is well-known that it does not depend on the choice of the reduced expression). We set

$$\begin{array}{rccc} \boldsymbol{\ell} : & W & \longrightarrow & \mathbb{Z}[\bar{S}] \\ & w & \longmapsto & \displaystyle\sum_{\omega \in \bar{S}} \ell_\omega(w)\omega. \end{array}$$

If Γ is an abelian group, a map $\varphi : W \to \Gamma$ is called a *weight function* if $\varphi(ww') = \varphi(w) + \varphi(w')$ for all w, $w' \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$ (see [18, §3.1]). The maps ℓ_ω (viewed as functions with values in $\mathbb{Z}$) and $\boldsymbol{\ell}$ are weight functions. In fact, the map $\boldsymbol{\ell}$ is universal in the following sense:

Lemma 1.1. *Let $\varphi : W \to \Gamma$ be a weight function. Then there exists a unique morphism of groups $\bar{\varphi} : \mathbb{Z}[\bar{S}] \to \Gamma$ such that $\varphi = \bar{\varphi} \circ \boldsymbol{\ell}$.*

Proof. Clear. $\square$

Let $\mathrm{Maps}(\bar{S}, \Gamma)$ be the set of maps $\bar{S} \to \Gamma$ and let $\mathrm{Weights}(W, \Gamma)$ be the set of weight functions $W \to \Gamma$. The lemma 1.1 shows that there are canonical bijections

$$\mathrm{Weights}(W, \Gamma) \overset{\sim}{\longleftrightarrow} \mathrm{Maps}(\bar{S}, \Gamma) \overset{\sim}{\longleftrightarrow} \mathrm{Hom}(\mathbb{Z}[\bar{S}], \Gamma).$$

We shall identify these three sets all along this paper. More precisely, we shall work with maps $\varphi : \bar{S} \to \Gamma$ that we shall see, according to our needs, as morphisms of groups $\mathbb{Z}[\bar{S}] \to \Gamma$ or as weight functions $W \to \Gamma$. In particular, we can talk as well about $\varphi(w)$ (for $w \in W$) as about $\varphi(\lambda)$ (for $\lambda \in \mathbb{Z}[\bar{S}]$): we hope that it will not lead to some confusion. For instance, $\mathrm{Ker}\,\varphi$ is a subgroup of $\mathbb{Z}[\bar{S}]$ (and not of W!).

1.B. **Hecke algebras with unequal parameters.** Let us first fix the notation that will be used throughout this paper.

Notation. *Let Γ be an abelian group and let $\varphi : \bar{S} \to \Gamma$ be a map.*

We shall use an exponential notation for the group algebra of Γ: $\mathbb{Z}[\Gamma] = \underset{\gamma \in \Gamma}{\oplus} \mathbb{Z}e^\gamma$, where $e^\gamma \cdot e^{\gamma'} = e^{\gamma+\gamma'}$ for all γ, $\gamma' \in \Gamma$. We then denote by $\mathscr{H}(W, S, \varphi)$ the *Hecke algebra with parameter* φ that is, the free $\mathbb{Z}[\Gamma]$-module with basis $(T_w)_{w \in W}$ endowed with the unique $\mathbb{Z}[\Gamma]$-bilinear associative multiplication completely determined by the following rules:

$$\begin{cases} T_w T_{w'} = T_{ww'} & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ (T_s - e^{\varphi(s)})(T_s + e^{-\varphi(s)}) = 0 & \text{if } s \in S. \end{cases}$$

If necessary, we denote by T_w^φ the element T_w to emphasize that it lives in the Hecke algebra with parameter φ.

This algebra is endowed with several involutions. We shall use only the following: if $\gamma \in \Gamma$ and $w \in W$, we set $\overline{e^\gamma} = e^{-\gamma}$ and $\overline{T}_w = T_{w^{-1}}^{-1}$ (note that T_w is invertible).

This extends by $\mathbb{Z}$-linearity to a $\mathbb{Z}[\Gamma]$-antilinear map $\mathscr{H}(W,S,\varphi) \to \mathscr{H}(W,S,\varphi)$, $h \mapsto \overline{h}$ which is an involutive antilinear automorphism of ring.

The previous construction is functorial. If $\rho : \Gamma \to \Gamma'$ is a morphism of abelian groups, then ρ induces a map

$$\rho_* : \mathscr{H}(W,S,\varphi) \longrightarrow \mathscr{H}(W,S,\rho\circ\varphi)$$

defined as the unique $\mathbb{Z}[\Gamma]$-linear map sending T_w^φ on $T_w^{\rho\circ\varphi}$ (here, $\mathscr{H}(W,S,\rho\circ\varphi)$ is viewed as a $\mathbb{Z}[\Gamma]$-algebra through the morphism of rings $\mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma']$ induced by ρ). It is then easily checked that

(1.2) ρ_* *is a morphism of* $\mathbb{Z}[\Gamma]$*-algebras.*

If $h \in \mathscr{H}(W,S,\varphi)$, then

(1.3) $$\overline{\rho_*(h)} = \rho_*(\overline{h}).$$

Moreover, if $\sigma : \Gamma' \to \Gamma''$ is another morphism of abelian groups, then

(1.4) $$(\sigma\circ\rho)_* = \sigma_* \circ \rho_*.$$

The next lemma is an obvious consequence:

Lemma 1.5. *The morphism of groups ρ is injective (respectively surjective, respectively bijective) if and only if the morphism of algebras ρ_* is.*

1.C. **Generic Hecke algebra.** Let R be the group algebra $\mathbb{Z}[\mathbb{Z}[\bar{S}]]$. Let $i : \bar{S} \to \mathbb{Z}[\bar{S}]$ be the canonical map. The Hecke algebra $\mathscr{H}(W,S,i)$ will then be denoted by $\mathscr{H}(W,S)$: it is called the *generic Hecke algebra.*

It is universal in the following sense: if we identify the map $\varphi : \bar{S} \to \Gamma$ with the morphism of groups $\varphi : \mathbb{Z}[\bar{S}] \to \Gamma$, then the Hecke algebra $\mathscr{H}(W,S,\varphi)$ is now equipped with a canonical morphism of R-algebras $\varphi_* : \mathscr{H}(W,S) \to \mathscr{H}(W,S,\varphi)$, so that $\mathscr{H}(W,S,\varphi)$ is a specialization of $\mathscr{H}(W,S)$, i.e.

$$\mathscr{H}(W,S,\varphi) = \mathbb{Z}[\Gamma] \otimes_R \mathscr{H}(W,S).$$

2. KAZHDAN-LUSZTIG CELLS

2.A. **Kazhdan-Lusztig basis.** To define the Kazhdan-Lusztig basis, we shall need to work all along this paper under the following hypothesis:

> **Hypothesis and notation.** *Until the end of this paper, we assume that Γ is endowed with a total order $\leqslant$ compatible with the group law. We denote respectively by $\Gamma_{>0}$, $\Gamma_{\geqslant 0}$, $\Gamma_{<0}$ and $\Gamma_{\leqslant 0}$ the set of positive, non-negative, negative, non-positive elements of Γ.*

If E is any subset of Γ, we set $\mathbb{Z}[E] = \bigoplus_{\gamma \in E} \mathbb{Z}e^\gamma$. For instance, $\mathbb{Z}[\Gamma_{\leqslant 0}]$ is a subring of $\mathbb{Z}[\Gamma]$ and $\mathbb{Z}[\Gamma_{<0}]$ is an ideal of $\mathbb{Z}[\Gamma_{\leqslant 0}]$. Let

$$\mathscr{H}_{<0}(W,S,\varphi) = \bigoplus_{w\in W} \mathbb{Z}[\Gamma_{<0}]\ T_w.$$

Then, if $w \in W$, there exists [18, Theorem 5.2] a unique element $C_w \in \mathscr{H}(W,S,\varphi)$ such that

$$\begin{cases} \overline{C}_w = C_w, \\ C_w \equiv T_w \mod \mathscr{H}_{<0}(W,S,\varphi). \end{cases}$$

Again, if necessary, the element C_w will be denoted by C_w^φ.

The family $(C_w)_{w\in W}$ is a $\mathbb{Z}[\Gamma]$-basis of $\mathscr{H}(W,S,\varphi)$ (called the *Kazhdan-Lusztig basis* [18, Theorem 5.2]). If x, $y \in W_n$, then we shall write $x \overset{L,\varphi}{\longleftarrow} y$ (respectively $x \overset{R,\varphi}{\longleftarrow} y$, respectively $x \overset{LR,\varphi}{\longleftarrow} y$) if there exists $h \in \mathscr{H}_n$ such that the coefficient of C_x in the decomposition of hC_y (respectively $C_y h$, respectively hC_y or $C_y h$) is non-zero. We denote by $\leqslant_L^\varphi$ (respectively $\leqslant_R^\varphi$, respectively $\leqslant_{LR}^\varphi$) the transitive closure of $\overset{L,\varphi}{\longleftarrow}$ (respectively $\overset{R,\varphi}{\longleftarrow}$, respectively $\overset{LR,\varphi}{\longleftarrow}$). Then $\leqslant_L^\varphi$, $\leqslant_R^\varphi$ and $\leqslant_{LR}^\varphi$ are preorders on W and we denote respectively by $\sim_L^\varphi$, $\sim_R^\varphi$ and $\sim_{LR}^\varphi$ the associated equivalence relations [18, Chapter 8]. An equivalence class for $\sim_L^\varphi$ (respectively $\sim_R^\varphi$, respectively $\sim_{LR}^\varphi$) is called a *left* (respectively *right*, respectively *two-sided*) *cell* (for (W,S,φ)). We recall the following result [18, §8.1]: if x, $y \in W_n$, then

(2.1) $$x \sim_L^\varphi y \Longleftrightarrow x^{-1} \sim_R^\varphi y^{-1}.$$

If $w \in W$, and if $? \in \{L, R, LR\}$, we set

$$\mathrm{Cell}_?^\varphi(w) = \{x \in W \mid x \sim_?^\varphi w\}.$$

2.B. **Cell representations.** Let $? \in \{L, R, LR\}$. For simplification, we define a ?-*ideal* to be a left ideal if $? = L$, a right ideal if $? = R$ and a two-sided ideal if $? = LR$. Similarly, if A is a ring, a ?-*module* (or an A-?-module) is a left A-module if $? = L$, a right A-module if $? = R$ and an (A, A)-bimodule if $? = LR$.

If C is a cell in W for $\sim_?^\varphi$, we set, following [18, §8.3],

$$\mathscr{H}(W,S,\varphi)_{\leqslant_?^\varphi C} = \bigoplus_{w \leqslant_?^\varphi C} \mathbb{Z}[\Gamma]\ C_w, \quad \mathscr{H}(W,S,\varphi)_{<_?^\varphi C} = \bigoplus_{w <_?^\varphi C} \mathbb{Z}[\Gamma]\ C_w$$

and $$M_C^{?,\varphi} = \mathscr{H}(W,S,\varphi)_{\leqslant_?^\varphi C} / \mathscr{H}(W,S,\varphi)_{<_?^\varphi C}.$$

Then, by definition, $\mathscr{H}(W,S,\varphi)_{\leqslant_?^\varphi C}$ and $\mathscr{H}(W,S,\varphi)_{<_?^\varphi C}$ are ?-ideals and $M_C^{?,\varphi}$ is an $\mathscr{H}(W,S,\varphi)$-?-module. Note that $M_C^{?,\varphi}$ is a free $\mathbb{Z}[\Gamma]$-module with basis the image of $(C_w)_{w\in C}$.

Let $\mathrm{aug} : \mathbb{Z}[\Gamma] \to \mathbb{Z}$ be the augmentation morphism and view $\mathbb{Z}$ (or $\mathbb{Q}$) as a $\mathbb{Z}[\Gamma]$-algebra through aug. We have an isomorphism of rings $\mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} \mathscr{H}(W,S,\varphi) \simeq \mathbb{Z}W$.

Let $\mathbb{Z}M_C^{?,\varphi}$ (respectively $\mathbb{Q}M_C^{?,\varphi}$) be the $\mathbb{Z}W$-?-module (respectively $\mathbb{Q}W$-?-module) $\mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} M_C^{?,\varphi}$ (respectively $\mathbb{Q} \otimes_{\mathbb{Z}[\Gamma]} M_C^{?,\varphi}$).

If W is *finite*, we denote by $\chi_C^{?,\varphi}$ the character of $\mathbb{Q}M_C^{?,\varphi}$ (if $? = LR$, it is a character of $W \times W$).

2.C. **Strictly increasing morphisms.** The main goal of this paper is to study how the partition into cells behaves whenever the datum $(\Gamma, \leqslant, \varphi)$ varies. The first easy remark is that this partition does not change by composition with a strictly increasing morphism of groups:

Proposition 2.2. *Let Γ' be a totally ordered abelian group, let $\rho : \Gamma \to \Gamma'$ be a strictly increasing morphism of groups and let $? \in \{L, R, LR\}$. Then:*

(a) *ρ_* is injective.*
(b) *If $w \in W$, then $\rho_*(C_w^\varphi) = C_w^{\rho\circ\varphi}$.*
(c) *The relations $\leqslant_?^\varphi$ et $\leqslant_?^{\rho\circ\varphi}$ coincide (as well as the relations $\sim_?^\varphi$ et $\sim_?^{\rho\circ\varphi}$).*
(d) *If $w \in W$, then $\mathrm{Cell}_?^\varphi(w) = \mathrm{Cell}_?^{\rho\circ\varphi}(w)$.*

Proof. The injectivity of ρ_* follows from the fact that ρ itself is injective (see Lemma 1.5). Hence (a) holds. Let us show (b). Let $C = \rho_*(C_w^\varphi)$. Then, by 1.3, we get $\overline{C} = C$. The fact that ρ is strictly increasing implies that $\rho_*(\mathbb{Z}[\Gamma_{<0}]) \subseteq \mathbb{Z}[\Gamma'_{<0}]$, so that

$$\rho_*(\mathscr{H}_{<0}(W, S, \varphi)) \subseteq \mathscr{H}_{<0}(W, S, \rho \circ \varphi). \tag{2.3}$$

Therefore, $C - T_w^{\rho\circ\varphi} \in \mathscr{H}_{<0}(W, S, \rho\circ\varphi)$. So $C = C_w^{\rho\circ\varphi}$ by the characterization of the elements of the Kazhdan-Lusztig basis.

Now (c) follows immediately from (a) and (b), while (d) is an immediate consequence of (c). $\square$

2.D. **Changing signs.** We shall show in this subsection that changing the sign of some values of the function φ has no effect on the partition into cells (and that it has little effect on the cell representations). For this, we shall work under the following hypothesis:

> **Hypothesis and notation.** *In this subsection, and only in this subsection, we fix a partition $S = S_+ \dot\cup S_-$ of S such that no element of S_+ is conjugate to an element of S_-. Let $\varphi' : \bar{S} \to \Gamma$ be the (well-defined) map such that*
>
> $$\varphi'(\bar{s}) = \begin{cases} \varphi(\bar{s}) & \text{if } s \in S_+, \\ -\varphi(\bar{s}) & \text{if } s \in S_-. \end{cases}$$

If $w \in W$, we set

$$\ell_\pm(w) = \sum_{\omega \in \bar{S}_\pm} \ell_\omega(w),$$

so that $\ell(w) = \ell_+(w) + \ell_-(w)$. Let $\theta : \mathscr{H}(W, S, \varphi) \longrightarrow \mathscr{H}(W, S, \varphi')$ be the unique $\mathbb{Z}[\Gamma]$-linear map such that

$$\theta(T_w^\varphi) = (-1)^{\ell_-(w)} T_w^{\varphi'}.$$

An elementary computation shows that:

Proposition 2.4. *The map $\theta : \mathscr{H}(W, S, \varphi) \longrightarrow \mathscr{H}(W, S, \varphi')$ is an isomorphism of $\mathbb{Z}[\Gamma]$-algebras. It sends $\mathscr{H}_{<0}(W, S, \varphi)$ onto $\mathscr{H}_{<0}(W, S, \varphi')$. If $h \in \mathscr{H}(W, S, \varphi)$, then $\theta(\overline{h}) = \overline{\theta(h)}$. Consequently, if $w \in W$, then*

$$\theta(C_w^\varphi) = (-1)^{\ell_-(w)} C_w^{\varphi'}.$$

Corollary 2.5. *If $? \in \{L, R, LR\}$, then the relations $\leqslant_?^\varphi$ and $\leqslant_?^{\varphi'}$ coincide. Similarly, the relations $\sim_?^\varphi$ and $\sim_?^{\varphi'}$ coincide.*

Let $\gamma : W \to \{1, -1\}$ denote the unique linear character of W such that

$$\gamma(s) = \begin{cases} 1 & \text{if } s \in S_+, \\ -1 & \text{if } s \in S_-. \end{cases}$$

In other words, $\gamma(w) = (-1)^{\ell_-(w)}$. Let $\mathbb{Z}_\gamma$ (respectively $\mathbb{Z}_{\gamma \boxtimes \gamma}$) denote the (left or right) $\mathbb{Z}W$-module (respectively the $(\mathbb{Z}W, \mathbb{Z}W)$-bimodule) of $\mathbb{Z}$-rank 1 on which W (respectively $W \times W$) acts through the linear character γ (respectively the linear character $\gamma \boxtimes \gamma : W \times W \to \{1, -1\}$, $(w, w') \mapsto \gamma(w)\gamma(w')$). If C is a cell for $\sim_?^\varphi$, then an easy computation using the isomorphism θ shows that

(2.6)
$$\mathbb{Z}M_C^{?,\varphi'} \simeq \begin{cases} \mathbb{Z}_\gamma \otimes_{\mathbb{Z}} \mathbb{Z}M_C^{?,\varphi} & \text{if } ? \in \{L, R\}, \\ \mathbb{Z}_{\gamma \boxtimes \gamma} \otimes_{\mathbb{Z}} \mathbb{Z}M_C^{?,\varphi} & \text{if } ? = LR. \end{cases}$$

2.E. **Zero values of $\boldsymbol{\varphi}$.** It follows from Corollary 2.5 that the computation of Kazhdan-Lusztig cells can be reduced to the case where φ has non-negative values. We shall study here what happens when φ vanishes on some elements of S. We shall work under the following hypothesis:

> **Hypothesis and notation.** *In this subsection, and only in this subsection, we fix a partition $S = I \dot\cup J$ of S such that no element of I is conjugate to an element of J. Let W_I denote the subgroup of W generated by I and we set*
>
> $$\widetilde{J} = \{wtw^{-1} \mid w \in W_I \text{ and } t \in J\}.$$

Let $\widetilde{W}$ be the subgroup of W generated by $\widetilde{J}$. We shall also assume that

$$(*) \qquad \text{if } s \in I, \text{ then } \varphi(s) = 0.$$

If $w \in W$, let $\ell_I(w)$ and $\ell_J(w)$ denote respectively the number of occurrences of elements of I and J in a reduced expression of w.

By a result of Gal [17, Proposition 2.1] (see also [3, Theorem 1.1]), $(\widetilde{W}, \widetilde{J})$ is a Coxeter system and

$$W = W_I \ltimes \widetilde{W} \tag{2.7}$$

We shall denote by $\tilde{\ell} : \widetilde{W} \to \mathbb{N}$ the length function on $\widetilde{W}$ with respect to $\widetilde{J}$. In fact [3, Corollary 1.3], if $w \in \widetilde{W}$, then

$$\tilde{\ell}(w) = \ell_J(w). \tag{2.8}$$

If $\tilde{t} \in \widetilde{J}$, we denote by $\nu(\tilde{t})$ the unique element of J such that $\tilde{t}$ is W_I-conjugate to $\nu(\tilde{t})$ (see [3, (1.3)]). We set

$$\tilde{\varphi}(\tilde{t}) = \varphi(\nu(\tilde{t})).$$

By [3, Corollary 1.4] that, if $\tilde{t}$ and $\tilde{t}'$ are two elements of $\widetilde{J}$ which are conjugate in $\widetilde{W}$, then

$$\tilde{\varphi}(\tilde{t}) = \tilde{\varphi}(\tilde{t}'). \tag{2.9}$$

This shows that we can define a Hecke $\mathbb{Z}[\Gamma]$-algebra $\mathscr{H}(\widetilde{W}, \widetilde{J}, \tilde{\varphi})$. The group W_I acts on $\widetilde{W}$ and stabilizes $\widetilde{J}$ and the map $\tilde{\varphi}$, so it acts naturally on the Hecke algebra $\mathscr{H}(\widetilde{W}, \widetilde{J}, \tilde{\varphi})$. We can then define the semidirect product of algebras

$$W_I \ltimes \mathscr{H}(\widetilde{W}, \widetilde{J}, \tilde{\varphi}).$$

It is a $\mathbb{Z}[\Gamma]$-algebra with $\mathbb{Z}[\Gamma]$-basis $(x \cdot T_w^{\tilde{\varphi}})_{x \in W_I, w \in \widetilde{W}}$. Finally, let

$$\widetilde{\mathscr{H}} = \bigoplus_{w \in \widetilde{W}} \mathbb{Z}[\Gamma]\, T_w^{\varphi} \quad \subseteq \mathscr{H}(W, S, \varphi).$$

Proposition 2.10. *Recall that $\varphi(s) = 0$ if $s \in I$. Then:*

(a) *$\widetilde{\mathscr{H}}$ is a sub-algebra of $\mathscr{H}(W, S, \varphi)$.*

(b) *The unique $\mathbb{Z}[\Gamma]$-linear map $\theta : W_I \ltimes \mathscr{H}(\widetilde{W}, \widetilde{J}, \tilde{\varphi}) \longrightarrow \mathscr{H}(W, S, \varphi)$ that sends $x \cdot T_w^{\tilde{\varphi}}$ on T_{xw}^{φ} ($x \in W_I$, $w \in \widetilde{W}$) is an isomorphism of $\mathbb{Z}[\Gamma]$-algebras. It sends $\mathscr{H}(\widetilde{W}, \widetilde{J}, \tilde{\varphi})$ isomorphically onto $\widetilde{\mathscr{H}}$.*

(c) *If $x \in W_I$ and $w \in W$, then $C_x^{\varphi} = T_x^{\varphi}$, $C_{xw}^{\varphi} = T_x^{\varphi} C_w^{\varphi}$, $C_{wx}^{\varphi} = C_w^{\varphi} T_x^{\varphi}$. If moreover $w \in \widetilde{W}$, then $\theta(C_w^{\tilde{\varphi}}) = C_w^{\varphi}$.*

Proof. Since $\varphi(I) = \{0\}$, we have, for all $s \in I$,

$$(T_s^{\varphi})^2 = 1 \quad \text{and} \quad \overline{T_s^{\varphi}} = T_s^{\varphi}.$$

So it follows by an easy induction argument that:

(2.11) *If $\ell_J(xy) = \ell_J(x) + \ell_J(y)$, then $T_x^\varphi T_y^\varphi = T_{xy}^\varphi$.*

In particular, if $x \in W_I$ and $w \in W$, then $T_x^\varphi T_w^\varphi = T_{xw}^\varphi$ et $T_w^\varphi T_x^\varphi = T_{wx}^\varphi$. We deduce that $C_x^\varphi = T_x^\varphi$, $C_{xw}^\varphi = T_x^\varphi C_w^\varphi$, $C_{wx}^\varphi = C_w^\varphi T_x^\varphi$. This proves the first assertion of (c).

Consequently, if $x \in W_I$ and $t \in J$, then

$$T_{xtx^{-1}}^\varphi = T_x^\varphi T_t^\varphi (T_x^\varphi)^{-1}$$

and so, if we set $\tilde{t} = xtx^{-1}$, we have

$$(T_{\tilde{t}}^\varphi - e^{\tilde{\varphi}(\tilde{t})})(T_{\tilde{t}}^\varphi + e^{-\tilde{\varphi}(\tilde{t})}) = 0.$$

To show (a) and (b), it only remains to show the following: if w and w' are two elements of $\widetilde{W}$ such that $\tilde{\ell}(ww') = \tilde{\ell}(w) + \tilde{\ell}(w')$ (here, $\tilde{\ell}$ denotes the length function on $\widetilde{W}$ associated to $\widetilde{J}$), then

$$T_w^\varphi T_{w'}^\varphi = T_{ww'}^\varphi.$$

But this follows from 2.8 and 2.11.

In order to show the last assertion of (c), it is sufficient to notice that, if $h \in \mathscr{H}(\widetilde{W}, \widetilde{J}, \tilde{\varphi})$, then $\theta(\overline{h}) = \overline{\theta(h)}$, and to use the characterization of the elements of the Kazhdan-Lusztig basis. $\square$

Corollary 2.12. *Let a, $b \in W_I$, x, $y \in \widetilde{W}$. Then:*

(a) *$ax \leqslant_L^\varphi by$ if and only if $x \leqslant_L^{\tilde{\varphi}} y$.*
(b) *$xa \leqslant_R^\varphi yb$ if and only if $x \leqslant_{LR}^{\tilde{\varphi}} y$.*
(c) *$ax \leqslant_{LR}^\varphi yb$ if and only if there exists $c \in W_I$ such that $x \leqslant_{LR}^{\tilde{\varphi}} cyc^{-1}$.*

Proof. (b) and (c) follow easily from (a). So let us prove (a). It is sufficient to show that $ax \overset{L,\varphi}{\longleftarrow} by$ if and only if $x \overset{L,\tilde{\varphi}}{\longleftarrow} y$.

First assume that $ax \overset{L,\varphi}{\longleftarrow} by$. Then there exists $w \in W$ such that C_{ax}^φ occurs in the decomposition of $T_w^\varphi C_{by}^\varphi$. Let us write $w = cz$, with $c \in W_I$ and $z \in \widetilde{W}$. Then

$$T_w^\varphi C_{by}^\varphi = T_{cb}^\varphi T_{b^{-1}zb}^\varphi C_y^\varphi.$$

This shows that $cb = a$ and that C_x^φ occurs in the decomposition of $T_{b^{-1}zb}^\varphi C_y^\varphi$. By applying the Proposition 2.10 (and the isomorphism θ), we get that $x \overset{L,\tilde{\varphi}}{\longleftarrow} y$.

Conversely, assume that $x \overset{L,\tilde{\varphi}}{\longleftarrow} y$. Then there exists $w \in \widetilde{W}$ such that $C_x^{\tilde{\varphi}}$ occurs in the decomposition of $T_w^{\tilde{\varphi}} C_y^{\tilde{\varphi}}$. By applying the Proposition 2.10 (and the isomorphism θ), we get that C_{ax}^φ occurs in the decomposition of $T_{awb^{-1}}^\varphi C_{by}^\varphi$. So $ax \overset{L,\varphi}{\longleftarrow} by$. $\square$

We can immediately deduce the following corollary:

Corollary 2.13. *Assume that $\varphi(I) = \{0\}$. Then the left (respectively right, respectively two-sided) cells for (W, S, φ) are of the form $W_I \cdot C$ (respectively $C \cdot W_I$, respectively $W_I \cdot C \cdot W_I$), where C is a left (respectively right, respectively two-sided) cell for $(\widetilde{W}, \widetilde{J}, \tilde{\varphi})$.*

At the level of cell representations, we get:

Corollary 2.14. *If C if a left cell for $(\widetilde{W}, \widetilde{J}, \tilde{\varphi})$, then*

$$\mathbb{Z}M_{W_I \cdot C}^{L,\varphi} \simeq \mathrm{Ind}_{\widetilde{W}}^{W} \mathbb{Z}M_C^{L,\tilde{\varphi}}.$$

Proof. This follows easily from the Proposition 2.10 (and its proof) and the Corollaries 2.12 and 2.13. $\square$

3. Conjectures about cells

> *From now on, and until the end of this paper, we assume that S is finite. We shall use the notion, notation and results of the Appendix: positive subsets of $\mathbb{Z}[\bar{S}]$, topology on $\mathscr{P}os(\mathbb{Z}[\bar{S}])$, hyperplane arrangements,* $\mathrm{Pos}(\varphi)$*, $\mathscr{U}(\lambda)$, $\mathscr{H}_\lambda$...*

As explained before, the main aim of this paper is to study the behaviour of the relations $\sim_?^\varphi$ as the datum $(\Gamma, \leqslant, \varphi)$ varies. The first step is to show that the relations $\leqslant_?^\varphi$ (and $\sim_?^\varphi$) depend only on the positive subset $\mathrm{Pos}(\varphi) \in \mathscr{P}os(\mathbb{Z}[\bar{S}])$ (see Corollary 3.2). This allows us to define relations $\leqslant_?^X$ and $\sim_?^X$ for $X \in \mathscr{P}os(\mathbb{Z}[\bar{S}])$. Our conjectures are then about the behaviour of the relations $\sim_?^X$ whenever X runs over $\mathscr{P}os(\mathbb{Z}[\bar{S}])$: this involves the topology of this set together with the notion of hyperplane arrangements (see the Appendix).

3.A. **Positive subsets of $\mathbb{Z}[\bar{S}]$.** Let X be a positive subset of $\mathbb{Z}[\bar{S}]$. Let $\Gamma_X = \mathbb{Z}[\bar{S}]/(X \cap (-X))$ and let $\leqslant_X$ be the total order on Γ_X defined in A.4. Let $\varphi_X : \bar{S} \to \Gamma_X$ be the canonical map (it is the composition of the natural map $\bar{S} \to \mathbb{Z}[\bar{S}]$ with the canonical map $\mathrm{can}_X : \mathbb{Z}[\bar{S}] \to \Gamma_X$ of the Appendix). For simplification, the relations $\leqslant_?^{\varphi_X}$ et $\sim_?^{\varphi_X}$ will be denoted by $\leqslant_?^X$ and $\sim_?^X$. Similarly, if $w \in W$, we will denote by $\mathrm{Cell}_?^X(w)$ the subset $\mathrm{Cell}_?^{\varphi_X}(w)$ and, if C is a ?-cell for (W, S, φ_X), we denote by $\mathbb{Z}M_C^{?,X}$ the $\mathbb{Z}W$-module $\mathbb{Z}M_C^{?,\varphi_X}$ (and, if W is *finite*, we denote by $\chi_C^{?,X}$ the character $\chi_C^{?,\varphi_X}$). The next proposition and its corollary show that the family $\big((\Gamma_X, \leqslant_X, \varphi_X)\big)_{X \in \mathscr{P}os(\mathbb{Z}[\bar{S}])}$ is essentially exhaustive.

Proposition 3.1. *Let $X = \mathrm{Pos}(\varphi)$. Then there exists a unique morphism of groups $\bar{\varphi} : \Gamma_X \to \Gamma$ such that $\varphi = \bar{\varphi} \circ \varphi_X$. This morphism $\bar{\varphi}$ is strictly increasing.*

Proof. Indeed, $\mathrm{Ker}\,\varphi = X \cap (-X)$ so the existence, the unicity and the injectivity of $\bar{\varphi}$ are clear. On the other hand, if γ, $\gamma' \in \Gamma_X$ are such that $\gamma \leqslant \gamma'$ and if $\lambda \in \mathbb{Z}[\bar{S}]$ is a representative of $\gamma' - \gamma$, then $\lambda \in X$ by A.3. So $\varphi(\lambda) \geqslant 0$. In other words $\bar{\varphi}(\gamma' - \gamma) \geqslant 0$ that is, $\bar{\varphi}(\gamma) \leqslant \bar{\varphi}(\gamma')$. So $\bar{\varphi}$ is increasing: the fact that it is strictly increasing follows from the injectivity. $\square$

Corollary 3.2. *Let $X = \mathrm{Pos}(\varphi)$ and let $? \in \{L, R, LR\}$. Then:*

(a) *The relations $\leqslant_?^\varphi$ and $\leqslant_?^X$ are equal (as well as the relations $\sim_?^\varphi$ and $\sim_?^X$).*
(b) *If $w \in W$, then $\mathrm{Cell}_?^\varphi(w) = \mathrm{Cell}_?^X(w)$.*
(c) *If $C = \mathrm{Cell}_L^\varphi(w)$, then $\mathbb{Z}M_C^\varphi \simeq \mathbb{Z}M_C^X$.*

Thanks to Corollary 3.2, studying the behaviour of the relations $\sim_?^\varphi$ as the weight function φ varies is equivalent to studying the behaviour of relations $\sim_?^X$ as X runs over $\mathscr{P}os(\mathbb{Z}[\bar{S}])$. The following remark can be useful for switching from one point of view to the other:

REMARK 3.3 - Let $\lambda \in \mathbb{Z}[\bar{S}]$ and $X \in \mathscr{P}os(\mathbb{Z}[\bar{S}])$. Then the following hold:

(a) $X \in \mathscr{U}(\lambda)$ if and only if $\varphi_X(\lambda) <_X 0$.
(b) $X \in \mathscr{H}_\lambda$ if and only if $\varphi_X(\lambda) = 0$.
(c) $X \in \overline{\mathscr{U}(\lambda)}$ if and only if $\varphi_X(\lambda) \leqslant_X 0$. □

In the same spirit, the Corollary 2.5 can be translated into the Proposition 3.4 below. We first need some notation. If $\omega \in \bar{S}$, we denote by τ_ω the $\mathbb{Z}$-linear symmetry on $\mathbb{Z}[\bar{S}]$ such that $\tau_\omega(\omega) = -\omega$ and $\tau_\omega(\omega') = \omega'$ if $\omega' \neq \omega$. It is an automorphism of $\mathbb{Z}[\bar{S}]$: it induces an homeomorphism τ_ω^* of $\mathscr{P}os(\mathbb{Z}[\bar{S}])$.

Proposition 3.4. *If $\omega \in \bar{S}$, if $X \in \mathscr{P}os(\mathbb{Z}[\bar{S}])$ and if $? \in \{L, R, LR\}$, then the relations $\sim_?^X$ and $\sim_?^{\tau_\omega^*(X)}$ coincide.*

Proof. The map τ_ω induces a strictly increasing isomorphism

$$\tau_\omega : \Gamma_{\tau_\omega^*(X)} \xrightarrow{\sim} \Gamma_X.$$

Therefore, the relations $\sim_?^{\varphi_{\tau_\omega^*(X)}}$ and $\sim_?^{\tau_\omega \circ \varphi_{\tau_\omega^*(X)}}$ coincide (see the Proposition 2.2). Now, let $\varphi'_X = \tau_\omega \circ \varphi_{\tau_\omega^*(X)} : \bar{S} \to \Gamma_X$. But

$$\varphi'_X(\omega') = \begin{cases} \varphi_X(\omega') & \text{if } \omega \neq \omega', \\ -\varphi_X(\omega) & \text{if } \omega = \omega'. \end{cases}$$

So the result follows from Corollary 2.5. □

3.B. **Conjectures.** A rational hyperplane arrangement $\mathfrak{A}$ in $\mathscr{P}os(\mathbb{Z}[\bar{S}])$ is called *complete* if $\mathscr{H}_\omega \in \mathfrak{A}$ for all $\omega \in \bar{S}$. Recall that

$$\begin{aligned}\mathscr{H}_\omega &= \{X \in \mathscr{P}os(\mathbb{Z}[\bar{S}]) \mid \omega \in X \cap -X\} \\ &= \{X \in \mathscr{P}os(\mathbb{Z}[\bar{S}]) \mid \varphi_X(\omega) = 0\}.\end{aligned}$$

If $\mathfrak{A}$ is a complete arrangement and if $\mathscr{F}$ is an $\mathfrak{A}$-facet, we denote by $W_\mathscr{F}$ the standard parabolic subgroup generated by $\bigcup_{\omega\in\bar{S},\mathscr{F}\subseteq\mathscr{H}_\omega} \omega$. As in the Appendix, we denote by $\mathrm{Fac}(\mathfrak{A})$ and $\mathrm{Cham}(\mathfrak{A})$ the set of $\mathfrak{A}$-facets and $\mathfrak{A}$-chambers respectively.

We denote by $\mathscr{R}el(W)$ the set of equivalence relations on W. If $\mathscr{R}$ and $\mathscr{R}'$ are two equivalence relations on W, we write $\mathscr{R} \preccurlyeq \mathscr{R}'$ if $\mathscr{R}$ is finer than $\mathscr{R}'$ (i.e. if $x\mathscr{R}y$ implies $x\mathscr{R}'y$). The poset $(\mathscr{R}el(W), \preccurlyeq)$ is a lattice (i.e. every pair of two elements has a supremum and an infimum): $\sup(\mathscr{R},\mathscr{R}')$ is the equivalence relation generated by $\mathscr{R}$ and $\mathscr{R}'$ while $\inf(\mathscr{R},\mathscr{R}')$ is the equivalence relation $\mathscr{R}''$ defined by $x\mathscr{R}''y$ if and only if $x\mathscr{R}y$ and $x\mathscr{R}'y$ (for x, $y \in W$). Finally, if $? \in \{L, R, LR\}$ and if H is a subgroup of W, we denote by $\mathrm{trans}_?^H$ the equivalence relation on W defined by the action of H (or $H \times H$) by translation on W:

$$x \ \mathrm{trans}_?^H \ y \Leftrightarrow \begin{cases} \exists\, w \in H,\ y = wx, & \text{if } ? = L, \\ \exists\, w \in H,\ y = xw, & \text{if } ? = R, \\ \exists\, w, w' \in H,\ y = wxw', & \text{if } ? = LR. \end{cases}$$

Conjecture A. *Assume that S is finite. Then there exists a finite complete rational hyperplane arrangement $\mathfrak{A}$ in $\mathscr{P}os(\mathbb{Z}[\bar{S}])$ satisfying the following properties (for all $? \in \{L, R, LR\}$):*

(a) *If X and Y are two positive subsets of $\mathbb{Z}[\bar{S}]$ belonging to the same $\mathfrak{A}$-facet $\mathscr{F}$, then the relations $\sim_?^X$ and $\sim_?^Y$ coincide (we will denote it by $\sim_?^\mathscr{F}$).*

(b) *Let $\mathscr{F}$ be an $\mathfrak{A}$-facet. Then the cells for the relation $\sim_?^\mathscr{F}$ are the minimal subsets C of W satisfying the following conditions:*

(b1) *For every chamber $\mathscr{C}$ such that $\mathscr{F} \subseteq \overline{\mathscr{C}}$, C is a union of cells for $\sim_?^\mathscr{C}$;*

(b2) *C is stable by translation by $W_\mathscr{F}$ (on the left if $? = L$, on the right if $? = R$, on the left and on the right if $? = LR$).*

REMARK 3.5 - With the notation of the above Conjecture, the statement (b) is equivalent to the following one:

(b′) *Let $\mathscr{F}$ be an $\mathfrak{A}$-facet. Then*

$$\sim_?^\mathscr{F} = \sup\Bigl(\mathrm{trans}_?^{W_\mathscr{F}}, \sup_{\substack{\mathscr{C}\in\mathrm{Cham}(\mathfrak{A}) \\ \mathscr{F}\subseteq\overline{\mathscr{C}}}} \sim_?^\mathscr{C}\Bigr).$$

Here, the suprema are computed in the set $\mathscr{R}el(W)$. □

REMARK 3.6 - The statement (b2) in the Conjecture A comes from Corollary 2.13. It is necessary, as it can already be seen in the case where $|S| = 1$. □

The previous remark (together with the Corollary A.16 of the Appendix) implies immediately the following proposition (which "justifies" the title of the paper):

Proposition 3.7. *If the Conjecture A holds for (W,S), then the map*

$$\begin{array}{ccc} \mathscr{P}os(\mathbb{Z}[\bar{S}]) & \longrightarrow & \mathscr{R}el(W) \\ X & \longmapsto & \sim_?^X \end{array}$$

is upper semicontinuous (for all $? \in \{L, R, LR\}$).

Corollary 3.8. *Let $? \in \{L, R, LR\}$, let $w \in W$ and let $\mathscr{P}(W)$ denote the power set of W. If the Conjecture A holds for (W,S), then the map*

$$\begin{array}{ccc} \mathscr{P}os(\mathbb{Z}[\bar{S}]) & \longrightarrow & \mathscr{P}(W) \\ X & \longmapsto & \mathrm{Cell}_?^X(w) \end{array}$$

is upper semicontinuous.

3.C. **Another form of Conjecture A.** Let us now give a translation of this conjecture in terms of maps $\varphi : \bar{S} \to \Gamma$. For this, let $\mathrm{SGN} = \{+, -, 0\}$ and let $\mathscr{E}$ be a finite set of elements in $\mathbb{Z}[\bar{S}] \setminus \{0\}$. Let $\mathrm{SGN}^{\mathscr{E}}$ denote the set of maps $\mathscr{E} \to \mathrm{SGN}$. If $X \in \mathscr{P}os(\mathbb{Z}[\bar{S}])$, we set:

$$\begin{array}{cccc} \mathrm{sgn}_{\mathscr{E}}(X) : & \mathscr{E} & \longrightarrow & \mathrm{SGN} \\ & \lambda & \longmapsto & \begin{cases} + & \text{if } \lambda \notin -X, \\ 0 & \text{if } \lambda \in X \cap (-X), \\ - & \text{if } \lambda \notin X. \end{cases} \end{array}$$

This defines a map

$$\mathrm{sgn}_{\mathscr{E}} : \mathscr{P}os(\mathbb{Z}[\bar{S}]) \longrightarrow \mathrm{SGN}\,.$$

Similarly, we set

$$\begin{array}{cccc} \mathrm{sgn}_{\mathscr{E}}(\varphi) : & \mathscr{E} & \longrightarrow & \mathrm{SGN} \\ & \lambda & \longmapsto & \begin{cases} + & \text{if } \varphi(\lambda) > 0, \\ 0 & \text{if } \varphi(\lambda) = 0, \\ - & \text{if } \varphi(\lambda) < 0. \end{cases} \end{array}$$

In other words,

$$\textbf{(3.9)} \qquad \mathrm{sgn}_{\mathscr{E}}(X) = \mathrm{sgn}_{\mathscr{E}}(\varphi_X) \quad \text{and} \quad \mathrm{sgn}_{\mathscr{E}}(\varphi) = \mathrm{sgn}_{\mathscr{E}}(\mathrm{Pos}(\varphi)).$$

Let $\mathfrak{A}_{\mathscr{E}}$ be the finite rational hyperplane arrangement $\{\mathscr{H}_\lambda \mid \lambda \in \mathscr{E}\}$. By the very definition of facets, the $\mathfrak{A}_{\mathscr{E}}$-facets are exactly the non-empty fibers of the map $\mathrm{sgn}_{\mathscr{E}} : \mathscr{P}os(\mathbb{Z}[\bar{S}]) \longrightarrow \mathrm{SGN}^{\mathscr{E}}$. If $\mathscr{F}$ is an $\mathfrak{A}_{\mathscr{E}}$-facet, then we denote by $\mathrm{sgn}_{\mathscr{E}}(\mathscr{F})$ the element $\mathrm{sgn}_{\mathscr{E}}(X) \in \mathrm{SGN}^{\mathscr{E}}$, where X is some (or any) element of $\mathscr{F}$.

We now endow SGN with the unique partial order $\preccurlyeq$ such that $0 \preccurlyeq +$, $0 \preccurlyeq -$ and $+$ and $-$ are not comparable. This defines a partial order on $\mathrm{SGN}^{\mathscr{E}}$ (componentwise) which is still denoted by $\preccurlyeq$. Then, by Proposition A.14, we have, for all facets $\mathscr{F}$ and $\mathscr{F}'$,

$$\textbf{(3.10)} \qquad \mathscr{F} \subseteq \overline{\mathscr{F}'} \textit{ if and only if } \mathrm{sgn}_{\mathscr{E}}(\mathscr{F}) \preccurlyeq \mathrm{sgn}_{\mathscr{E}}(\mathscr{F}').$$

(Note that this implies, by Corollary A.16, that the map $\mathrm{sgn}_{\mathscr{E}}$ is lower semicontinuous.) Finally, we say that the map φ is *$\mathscr{E}$-open* if $\mathrm{Pos}(\varphi)$ belongs to an $\mathfrak{A}_{\mathscr{E}}$-chamber: equivalently, φ is $\mathscr{E}$-open if $\mathrm{sgn}_{\mathscr{E}}(\varphi)(\lambda) \neq 0$ for all $\lambda \in \mathscr{E} \setminus \{0\}$.

By the above discussion, we get that the next Conjecture is clearly equivalent to Conjecture A (here, we denote by W_φ the parabolic subgroup generated by $\{s \in S \mid \varphi(s) = 0\}$).

Conjecture A'. *Assume that S is finite. Then there exists a finite set $\mathscr{E}$ in $\mathbb{Z}[\bar{S}]\backslash\{0\}$ containing $\bar{S}$ and such that, for all $? \in \{L, R, LR\}$, we have:*

(a) *If Γ and Γ' are two abelian ordered groups and if $\varphi : \bar{S} \to \Gamma$ and $\varphi' : \bar{S} \to \Gamma'$ are two maps such that $\mathrm{sgn}_{\mathscr{E}}(\varphi) = \mathrm{sgn}_{\mathscr{E}}(\varphi')$, then the relations $\sim_?^{\varphi}$ and $\sim_?^{\varphi'}$ coincide.*

(b) *If Γ is a totally ordered abelian group and if $\varphi : \bar{S} \to \Gamma$ is a map, then the cells for $\sim_?^{\varphi}$ are the minimal subsets C of W such that:*
   - (b1) *For all totally ordered abelian group Γ' and for all $\mathscr{E}$-open maps $\varphi' : \bar{S} \to \Gamma'$ such that $\mathrm{sgn}_{\mathscr{E}}(\varphi) \preccurlyeq \mathrm{sgn}_{\mathscr{E}}(\varphi')$, C is a union of cells for $\sim_?^{\varphi'}$.*
   - (b2) *C is stable by translation by W_φ (on the left if $? = L$, on the right if $? = R$, on the left and on the right if $? = LR$).*

An element $\lambda \in \mathbb{Z}[\bar{S}]$ is called *reduced* if $\lambda \neq 0$ and $\mathbb{Z}[\bar{S}]/\mathbb{Z}\lambda$ is torsion-free. If $\mathscr{H}$ is a rational hyperplane, then there exist only two reduced elements $\lambda \in \mathbb{Z}[\bar{S}]$ such that $\mathscr{H} = \mathscr{H}_\lambda$ (one is the opposite of the other). A subset $\mathscr{E}$ of $\mathbb{Z}[\bar{S}]$ is called *reduced* if all its elements are reduced. It is called *complete* if $\bar{S} \subseteq \mathscr{E}$. It is called *symmetric* if $\mathscr{E} = -\mathscr{E}$. If $\mathfrak{A}$ is a rational hyperplane arrangement, then there exists a unique reduced symmetric subset $\mathscr{E}$ of $\mathbb{Z}[\bar{S}]$ such that $\mathfrak{A} = \mathfrak{A}_{\mathscr{E}}$. In this case, $\mathfrak{A}$ is complete if and only if $\mathscr{E}$ is.

3.D. **Essential hyperplanes.** The Remark 3.5 implies easily the following

Proposition 3.11. *If the Conjecture A holds for (W, S) and two finite complete rational hyperplane arrangements $\mathfrak{A}$ and $\mathfrak{A}'$ of $\mathscr{P}os(\mathbb{Z}[\bar{S}])$, then it holds for (W, S) and for the finite complete arrangement $\mathfrak{A} \cap \mathfrak{A}'$.*

Proof. Clear. □

The Proposition 3.11 shows that, if the Conjecture A holds for (W, S), there exists a unique minimal finite complete rational hyperplane arrangement $\mathfrak{A}$ such that the statements (a) and (b) of Conjecture A hold. We call the elements of this minimal arrangement the *essential hyperplanes* of (W, S): indeed, if (W, S) is finite, they should be the same as the *essential hyperplanes* defined by M. Chlouveraki [9,

§4.3.1], which appear when she studied the Rouquier blocks of cyclotomic Hecke algebras associated to complex reflection groups.

Similarly, the Proposition 3.11 shows that, if the Conjecture A' holds for (W, S), there exists a unique minimal finite complete symmetric reduced subset $\mathscr{E}$ of $\mathbb{Z}[\bar{S}]$ such that the statements (a) and (b) of Conjecture A' holds: they will be called the *essential* elements of (W, S).

REMARK 3.12 - The Proposition 3.4 shows that, if the Conjecture A holds for (W, S), then the set of essential hyperplanes (respectively the set of essential elements) of (W, S) is stable by the action of all the symmetries τ_ω^* (respectively τ_ω), $\omega \in \bar{S}$. □

REMARK 3.13 - Let I be a subset of S. If s, $t \in I$, we shall write $s \sim_I t$ if s and t are conjugate in $W_I = \langle I \rangle$. We set $\bar{I} = I/{\sim_I}$. Note that $\bar{I}$ is not necessarily the image of I in $\bar{S}$. Nevertheless, the inclusion $I \hookrightarrow S$ induces a map $\bar{I} \to \bar{S}$, which extends by linearity to a map $\gamma_I : \mathbb{Z}[\bar{I}] \to \mathbb{Z}[\bar{S}]$. By functoriality (see the Appendix), this induces a map $\gamma_I^* : \mathscr{P}\!os(\mathbb{Z}[\bar{S}]) \to \mathscr{P}\!os(\mathbb{Z}[\bar{S}])$. Assume here that Conjecture A holds for (W, S) and (W_I, I): let $\mathfrak{A}$ (respectively $\mathfrak{A}_I$) denote the set of essential hyperplanes for (W, S) (respectively (W_I, I)). Then, since any left cell of W_I is the intersection of a left cell of W with W_I (see [10]), we get that $(\gamma_I^*)^{-1}(\mathfrak{A}_I)$ is contained in $\mathfrak{A} \cup \{\mathscr{P}\!os(\mathbb{Z}[\bar{S}])\}$.

Equivalently, if $\mathscr{E}$ and $\mathscr{E}_I$ denote the sets of essential elements of (W, S) and (W_I, I) respectively, then $\gamma_I(\mathscr{E}_I) \subseteq \mathscr{E} \cup \{0\}$. □

EXAMPLE 3.14 - If $|\bar{S}| = 1$, then Conjecture A is obviously true: the set of essential hyperplanes is $\{\mathscr{H}_S\}$ (note that $\bar{S} = \{S\}$) and the set of essential elements is $\{S, -S\} \subseteq \mathbb{Z}[\bar{S}]$. Indeed, if $\varphi : \bar{S} \to \Gamma$ and $\varphi' : \bar{S} \to \Gamma'$ are two maps such that $\varphi(S) > 0$ and $\varphi'(S) > 0$, then the relations $\sim_?^{\varphi}$ and $\sim_?^{\varphi'}$ coincide. On the other hand, if $\varphi(S) = 0$, then W contains only one cell for $\sim_?^{\varphi}$, namely W itself: it is clearly the smallest subset of W which is stable by translation by $W_\varphi = W$. □

3.E. **Example: the case where $|\bar{S}| = 2$.** Since the Conjecture A is expressed in terms of the topology of the set $\mathscr{P}\!os(\mathbb{Z}[\bar{S}])$, it might be difficult to understand it concretely. The purpose of this example is to give a concrete version of this statement whenever $|\bar{S}| = 2$: this will show that Conjecture A contains the Conjecture 0 of the introduction, and gives some precision for the case where φ vanishes at some simple reflections.

So assume here that $|\bar{S}| = 2$ and write $\bar{S} = \{\omega_1, \omega_2\}$. We shall identify $\mathbb{Z}[\bar{S}]$ with $\mathbb{Z}^2$ (through $(\lambda, \mu) \mapsto \lambda\omega_1 + \mu\omega_2$). If r is a rational number, we shall denote by $\mathscr{H}_r$ the hyperplane $\mathscr{H}_{-dr,d}$, where d is a non-zero natural number such that $dr \in \mathbb{Z}$. Then $\mathscr{H}_r = \{\mathbb{Z}[\bar{S}], X_r^+, X_r^-\}$, where

$$X_r^+ = \{(\lambda, \mu) \in \mathbb{Z}^2 \mid \lambda + r\mu \geqslant 0\} \quad \text{and} \quad X_r^- = \{(\lambda, \mu) \in \mathbb{Z}^2 \mid \lambda + r\mu \leqslant 0\}.$$

We set $\mathscr{H}_\infty = \mathscr{H}_{(1,0)}$. Then $\mathscr{H}_\infty = \{\mathbb{Z}[\bar{S}], X_\infty^+, X_\infty^-\}$, where

$$X_\infty^+ = \{(\lambda,\mu) \in \mathbb{Z}^2 \mid \mu \geqslant 0\} \quad \text{and} \qquad X_\infty^- = \{(\lambda,\mu) \in \mathbb{Z}^2 \mid \mu \leqslant 0\}.$$

Now, since Γ is torsion-free, the natural map $\Gamma \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ is injective, so we shall view Γ as embedded in the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$: in particular, if $r \in \mathbb{Q}$ and $\gamma \in \Gamma$, then $r\gamma$ is well-defined. Moreover, the order on Γ extends uniquely to a total order on $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ that we still denote by $\leqslant$. Now, let

$$\varphi(\omega_1) = a \quad \text{and} \quad \varphi(\omega_2) = b.$$

Then, one can immediately translate in concrete terms the fact that $\mathrm{Pos}(\varphi)$ belongs or not to one of these hyperplanes:

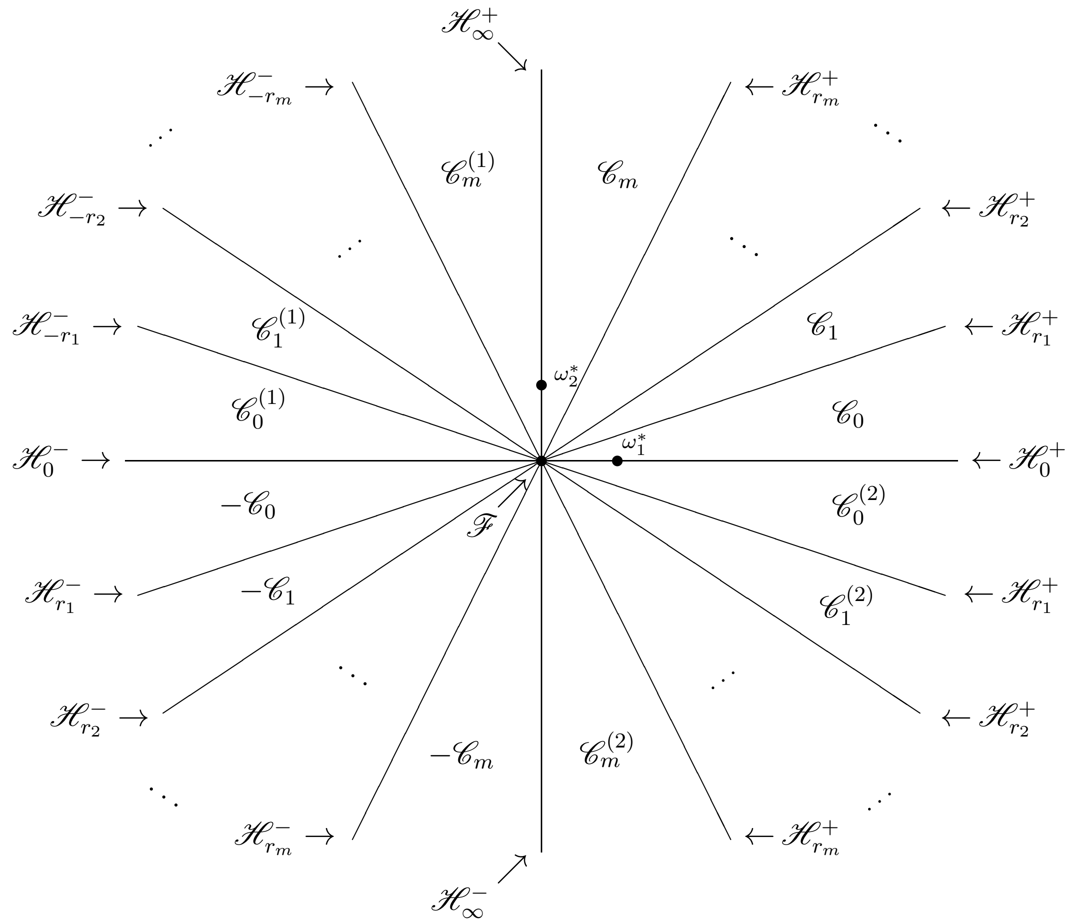
Lemma 3.15. *With the above notation, we have:*

(a) $\mathrm{Pos}(\varphi) \in \mathscr{H}_r$ *(respectively* $\mathrm{Pos}(\varphi) \in \mathscr{H}_\infty$*) if and only if* $b = ra$ *(respectively* $a = 0$*).*
(b) $\mathrm{Pos}(\varphi) = X_r^+$ *(respectively* X_r^-*) if and only if* $b = ra$ *and* $a > 0$ *(respectively* $a < 0$*).*
(c) $\mathrm{Pos}(\varphi) = X_\infty^+$ *(respectively* X_∞^-*) if and only if* $a = 0$ *and* $b > 0$ *(respectively* $b < 0$*).*

For simplification, we set $\tau_i = \tau_{\omega_i}$. Now, let $\mathfrak{A}$ be a finite complete rational hyperplane arrangement in $\mathscr{P}os(\mathbb{Z}[\bar{S}])$ which is stable under the actions of τ_1 and τ_2. Since $\tau_1 \circ \tau_2 = -\mathrm{Id}_{\mathbb{Z}[\bar{S}]}$, this is equivalent to say that it is stable under τ_1. Since all hyperplanes of $\mathscr{P}os(\mathbb{Z}[\bar{S}])$ are of the form $\mathscr{H}_r$ for $r \in \mathbb{Q} \cup \{\infty\}$, and since $\tau_1(\mathscr{H}_r) = \mathscr{H}_{-r}$ if $r \in \mathbb{Q}$ (and $\tau_1(\mathscr{H}_\infty) = \mathscr{H}_\infty$), there exist positive rational numbers $0 < r_1 < r_2 < \cdots < r_m$ such that

$$\mathfrak{A} = \{\mathscr{H}_0, \mathscr{H}_{r_1}, \mathscr{H}_{-r_1}, \mathscr{H}_{r_2}, \mathscr{H}_{-r_2}, \ldots, \mathscr{H}_{r_m}, \mathscr{H}_{-r_m}, \mathscr{H}_\infty\}.$$

Let us draw $\mathrm{Pos}^{-1}(\mathfrak{A})$ in $\mathbb{R}[\bar{S}]^* = \mathbb{R}\omega_1^* \oplus \mathbb{R}\omega_2^*$, where (ω_1^*, ω_2^*) is the dual basis of (ω_1, ω_2):



In this figure, we have written $\mathscr{H}_r^\pm = \{X_r^\pm\}$ and we have also drawn all the $\mathfrak{A}$-facets (or at least their preimage under Pos): note that $\mathscr{C}_i^{(j)} = \tau_j(\mathscr{C}_i)$ (for $0 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant 2$) and that we have the following property (where $r_0 = 0$ and $r_{m+1} = \infty$):

Lemma 3.16. *Let $i \in \{0, 1, 2, \dots, m-1, m\}$. Then $\mathrm{Pos}(\varphi) \in \mathscr{C}_i$ if and only if $r_i a < b < r_{i+1} a$ and $a > 0$.*

Note also that, if $1 \leqslant i \leqslant m$, then $\mathscr{C}_i$ and $\mathscr{C}_{i+1}$ are the only chambers $\mathscr{C}$ such that $\mathscr{H}_{r_i}^+ \subseteq \overline{\mathscr{C}}$, that $\mathscr{C}_0$ and $\mathscr{C}_0^{(2)}$ are the only chambers $\mathscr{C}$ such that $\mathscr{H}_0^+ \subseteq \overline{\mathscr{C}}$ and that $\mathscr{C}_m$ and $\mathscr{C}_m^{(1)}$ are the only chambers $\mathscr{C}$ such that $\mathscr{H}_\infty^+ \subseteq \overline{\mathscr{C}}$. All these descriptions (together with Lemma 3.4) show that, whenever $|\bar{S}| = 2$, Conjecture A is equivalent to the following one (the statements (a) and (b) are the same as the statements (a) and (b) in the Conjecture 0 of the introduction: the statements (c) and (d) gives an extra-information in the case where some parameters are equal to zero):

Conjecture A". *Assume that* $|\bar{S}| = 2$ *and let* $? \in \{L, R, LR\}$. *Then there exist rational numbers* $0 < r_1 < \cdots < r_m$ *(depending only on* (W, S)*) such that (setting* $r_0 = 0$ *and* $r_{m+1} = +\infty$*), if* Γ *and* Γ' *are two totally ordered abelian groups and* $\varphi : \bar{S} \to \Gamma$ *and* $\varphi' : \bar{S} \to \Gamma'$ *are maps, then:*

(a) *If* $0 \leqslant i \leqslant m$ *and if* $\mathrm{Pos}(\varphi)$ *and* $\mathrm{Pos}(\varphi')$ *are in* $\mathscr{C}_i$, *then the relations* $\sim_?^{\varphi}$ *and* $\sim_?^{\varphi'}$ *coincide. We denote it by* $\sim_?^{\mathscr{C}_i}$.

(b) *If* $1 \leqslant i \leqslant m$, *then* $\sim_?^{X_{r_i}^+}$ *is generated by* $\sim_?^{\mathscr{C}_{i-1}}$ *and* $\sim_?^{\mathscr{C}_i}$.

(c) *The relation* $\sim_?^{X_0^+}$ *is generated by* $\sim_?^{\mathscr{C}_0}$ *and (left, right or two-sided) translation by* W_{ω_2}.

(d) *The relation* $\sim_?^{X_\infty^+}$ *is generated by* $\sim_?^{\mathscr{C}_m}$ *and (left, right or two-sided) translation by* W_{ω_1}.

4. Conjectures about cell representations

It seems reasonable to expect that the Conjecture A is compatible with the construction of cell representations. We shall make this more precise here.

> **Hypothesis and notation.** *In this section, and only in this section, we assume that* S *is finite and that the Conjecture A holds for* (W, S). *We denote by* $\mathfrak{A}$ *the set of corresponding essential hyperplanes.*

Let $X \in \mathscr{P}os(\mathbb{Z}[\bar{S}])$, let $? \in \{L, R, LR\}$ and let C be a cell for $\sim_?^X$. Let $\mathscr{C}$ be an $\mathfrak{A}$-chamber such that $X \in \overline{\mathscr{C}}$ and let $Y \in \mathscr{C}$. Since the Conjecture A holds, we have $C = \dot{\bigcup}_{i \in \mathscr{I}} C_i$, where the C_i's are cells for $\sim_?^Y$.

Conjecture B. *There exist a natural number* d *and a partition* $\mathscr{I} = \dot{\bigcup}_{1 \leqslant k \leqslant d} \mathscr{I}_k$ *satisfying the following properties:*

(a) *If* $i \in \mathscr{I}_k$ *and* $j \in \mathscr{I}_l$ *are such that* $C_i \leqslant_?^Y C_j$, *then* $k \leqslant l$.

(b) *There exists a filtration* $M_0 = 0 \subseteq M_1 \subseteq \cdots \subseteq M_d = \mathbb{Z}M_C^X$ *of the* $\mathbb{Z}W$*-?-module* $\mathbb{Z}M_C^{?,X}$ *such that*

$$M_k/M_{k-1} \simeq \bigoplus_{i \in \mathscr{I}_k} \mathbb{Z}M_{C_i}^{?,Y}.$$

REMARK - In the above Conjecture, the "new" statement is the statement (b). Indeed, the partition of $\mathscr{I}$ satisfying (a) can be obtained by taking the fibers of Lusztig's **a**-function, if we assume that Lusztig's Conjectures [18, Conjecture 13.4, Conjecture 13.12 (a) and Conjectures **P1-P15** in Chapter 14] hold (indeed, in this case, the **a**-function would take only finitely many values). □

A much weaker version is given by:

Conjecture B⁻. *Assume that W is finite. Then*

$$\chi_C^{?,X} = \sum_{i \in \mathscr{I}} \chi_{C_i}^{?,Y}.$$

5. EXAMPLES

We shall illustrate here Conjectures A and B by several examples. Most of the computer calcutations that have lead to some of the results of this section were done by using M. Geck's programs (using the package `chevie` of `GAP3` [12]): we thank him warmly for his help.

5.A. **Finite dihedral groups.** Assume in this subsection, and only in this subsection, that $S = \{s,t\}$ and that st has finite even order $2m$ with $m \geqslant 2$. So we can identify S and $\bar{S}$. Let $w_0 = (st)^m = (ts)^m$ be the longest element W: it is central. If $w \in W$, we set $\mathscr{R}(w) = \{u \in S \mid wu < w\}$. Let

$$\Lambda_s = \{w \in W \mid \mathscr{R}(w) = \{s\}\} \quad \text{et} \quad \Lambda_t = \{w \in W \mid \mathscr{R}(w) = \{t\}\}.$$

An easy computation [18, §8.7] shows that the partition of W into left cells for (W,S,φ) is given by the following table (whenever φ has non-negative values):

φ	Left cells
$0 = \varphi(s) = \varphi(t)$	W
$0 = \varphi(s) < \varphi(t)$	$\{1,s\},\ \Lambda_s \setminus \{s\},\ \Lambda_t \setminus \{sw_0\},\ \{sw_0, w_0\}$
$0 < \varphi(s) < \varphi(t)$	$\{1\},\ \{s\},\ \Lambda_s \setminus \{s\},\ \Lambda_t \setminus \{sw_0\},\ \{sw_0\},\ \{w_0\}$
$0 < \varphi(s) = \varphi(t)$	$\{1\},\ \Lambda_s,\ \Lambda_t,\ \{w_0\}$
$0 < \varphi(t) < \varphi(s)$	$\{1\},\ \{t\},\ \Lambda_s \setminus \{tw_0\},\ \Lambda_t \setminus \{t\},\ \{tw_0\},\ \{w_0\}$
$0 = \varphi(t) < \varphi(s)$	$\{1,t\},\ \Lambda_s \setminus \{tw_0\},\ \Lambda_t \setminus \{t\},\ \{tw_0, w_0\}$

Proposition 5.1. *The Conjectures A and B hold if $|S| = 2$ and $|W| < \infty$. The essential hyperplanes are $\mathscr{H}_s$, $\mathscr{H}_t$, $\mathscr{H}_{s-t}$ and $\mathscr{H}_{s+t}$.*

Proof. Let us first show Conjecture A. By the discussion of §3.E, it is sufficient to show Conjecture A". For this, take $m = 1$ and $r_1 = 1$: then the statements (a), (b), (c) and (d) are easily checked by inspection of the above table.

Let us now show Conjecture B. There are finitely many cases to be considered: using the automorphism of the Coxeter graph of (W, S), the number of cases can be drastically limited. We shall only consider the following one, the other ones being treated similarly. Let $\varphi : \bar{S} \to \Gamma$, $\varphi' : \bar{S} \to \Gamma'$ and $\varphi'' : \bar{S} \to \Gamma''$ be maps such that

$$\varphi(s) = \varphi(t) > 0, \quad \varphi'(s) > \varphi'(t) > 0 \quad \text{and} \quad \varphi''(t) > \varphi''(s) > 0.$$

Here, Γ, Γ' and Γ'' are totally ordered abelian groups. We also denote by $\mathrm{aug} : \mathscr{H}(W, S, \varphi) \to \mathbb{Z}W$, $\mathrm{aug}' : \mathscr{H}(W, S, \varphi') \to \mathbb{Z}W$ and $\mathrm{aug}'' : \mathscr{H}(W, S, \varphi'') \to \mathbb{Z}W$ the morphisms of rings induced respectively by the augmentation morphisms $\mathbb{Z}[\Gamma] \to \mathbb{Z}$, $\mathbb{Z}[\Gamma'] \to \mathbb{Z}$ and $\mathbb{Z}[\Gamma''] \to \mathbb{Z}$. If $w \in W$, we set

$$G_w = \mathrm{aug}(C_w^{\varphi}), \quad G'_w = \mathrm{aug}'(C_w^{\varphi'}) \quad \text{and} \quad G''_w = \mathrm{aug}''(C_w^{\varphi''}).$$

Let $C = \Lambda_s$: it is a left cell for (W, S, φ) and, by the Table above (and easy computations), we have that:

($\mathscr{C}'$) $C \setminus \{tw_0\}$ and $\{tw_0\}$ are left cells for (W, S, φ') and $\{tw_0\} <_L^{\varphi'} C \setminus \{tw_0\}$.

($\mathscr{C}''$) $\{s\}$ and $C \setminus \{s\}$ are left cells for (W, S, φ'') and $C \setminus \{s\} <_L^{\varphi''} \{s\}$.

Note that $G_{w_0} = G'_{w_0} = G''_{w_0} = \sum_{w \in W} w$. We set

$$I = \bigoplus_{\substack{w \in W \\ ws < w}} \mathbb{Z}G_w, \quad I' = \bigoplus_{\substack{w \in W \\ ws < w}} \mathbb{Z}G'_w \quad \text{and} \quad I'' = \bigoplus_{\substack{w \in W \\ ws < w}} \mathbb{Z}G''_w.$$

Then I, I' and I'' are left ideals of $\mathbb{Z}W$ (by [18, Lemma 8.6]) and

$$I = I' = I''.$$

Indeed, this last equality can be proved by tedious computations using formulas given by Lusztig [18, Propositions 7.3 and 7.6], but it might also be proved by using [18, Lemma 8.4]: this last lemma shows that, since $G_s = G'_s = G''_s = s + 1$,

$$I = I' = I'' = \{h \in \mathbb{Z}W \mid h(s+1) = 2h\} = \{h \in \mathbb{Z}W \mid hs = h\}.$$

Now, by definition,

$$\mathbb{Z}M_C^{L,\varphi} \simeq I/\mathbb{Z}G_{w_0}.$$

Now, let

$$M' = \mathbb{Z}G'_{tw_0} \oplus \mathbb{Z}G'_{w_0} \quad \text{and} \quad M'' = \bigoplus_{w \in (C \cup \{w_0\}) \setminus \{s\}} \mathbb{Z}G''_w.$$

Then

$$M'_0 = 0 \subseteq M'_1 = M'/\mathbb{Z}G_{w_0} \subseteq M'_2 = I'/\mathbb{Z}G_{w_0} = \mathbb{Z}M_C^{L,\varphi}$$

and
$$M''_0 = 0 \subseteq M''_1 = M''/\mathbb{Z}G_{w_0} \subseteq M''_2 = I''/\mathbb{Z}G_{w_0} = \mathbb{Z}M_C^{L,\varphi}$$

are two filtrations of $\mathbb{Z}M_C^{L,\varphi}$ by left $\mathbb{Z}W$-modules. Moreover, by definition,

- $M'_1/M'_0 \simeq \mathbb{Z}M_{\{tw_0\}}^{L,\varphi'}$ and $M'_2/M'_1 \simeq \mathbb{Z}M_{C \setminus \{tw_0\}}^{L,\varphi'}$.

- $M_1''/M_0'' \simeq \mathbb{Z}M_{C\setminus\{s\}}^{L,\varphi''}$ and $M_2''/M_1'' \simeq \mathbb{Z}M_{\{s\}}^{L,\varphi''}$.

This shows Conjecture B in this particular case, taking into account the statements $(\mathscr{C}')$ and $(\mathscr{C}'')$ above. □

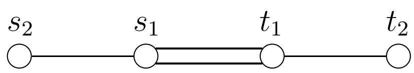
5.B. **Infinite dihedral group.** The same argument as in the finite case (see Proposition 5.1) shows that the following proposition holds:

Proposition 5.2. *If $S = \{s,t\}$ and st has infinite order, then the Conjectures A and B hold. The essential hyperplanes of (W,S) are $\mathscr{H}_s$, $\mathscr{H}_t$, $\mathscr{H}_{s-t}$ and $\mathscr{H}_{s+t}$.*

For general (W,S), the Remark 3.13 together with the Propositions 5.1 and 5.2 shows immediately that:

Proposition 5.3. *Assume that Conjecture A holds for (W,S). Let s and t be two elements of S such that $\bar{s} \neq \bar{t}$ and such that the order of st is greater than 3. Then $\mathscr{H}_{\bar{s}-\bar{t}}$ and $\mathscr{H}_{\bar{s}+\bar{t}}$ are essential hyperplanes of (W,S). Equivalently, $\bar{s}-\bar{t}$ and $\bar{s}+\bar{t}$ are essential elements of (W,S).*

5.C. **Type F_4.** Assume in this subsection, and only in this subsection, that (W,S) is of type F_4. Let $S = \{s_1, s_2, t_1, t_2\}$ be indexed in such a way that the Coxeter graph of (W,S) is



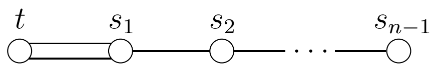
Let $s = \{s_1, s_2\}$ and $t = \{t_1, t_2\}$, so that $\bar{S} = \{s,t\}$.

Theorem 5.4 (Geck)**.** *If (W,S) is of type F_4, then the Conjectures A and B$^-$ hold for (W,S). With the above notation, the essential hyperplanes of (W,S) are $\mathscr{H}_s$, $\mathscr{H}_t$, $\mathscr{H}_{s-2t}$, $\mathscr{H}_{s-t}$, $\mathscr{H}_{2s-t}$, $\mathscr{H}_{s+2t}$, $\mathscr{H}_{s+t}$ and $\mathscr{H}_{2s+t}$.*

REMARK - With the notation of §3.E, this is equivalent to say that Conjecture A" holds and that we can take $m = 3$, $r_1 = 1/2$, $r_2 = 1$ and $r_3 = 2$. □

Proof. In [11], M. Geck has computed the Kazhdan-Lusztig cells W for all choices of map φ so that $\varphi(S) \subset \Gamma_{>0}$: this involves both theoretical and computational (using `GAP3`) arguments. To get the cells whenever $\varphi(S) \subset \Gamma_{\geqslant 0}$, it is then sufficient to use Corollary 2.13 (and the decomposition $W = \mathfrak{S}_3 \ltimes W(D_4)$, see 2.7) together with the knowledge of the cells in type D_4 (which can be obtained again by using `GAP3`). Using these results, one can easily check Conjecture A" (so that Conjecture A holds). Conjecture B$^-$ is then also checked by computations using `GAP3`. Note that all these computations are simplified thanks to the involutive automorphism of (W,S) that sends s_i to t_i. $\square$

5.D. **Type *B*.** Assume in this subsection, and only in this subsection, that (W,S) is the pair (W_n,S_n), where W_n is of type B_n ($n \geqslant 2$), that $S_n = \{t, s_1, s_2, \dots, s_{n-1}\}$, and that the Coxeter graph of (W_n,S_n) is

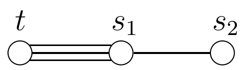


Identify t and $\{t\}$ and let $s = \{s_1, s_2, \dots, s_{n-1}\}$. So $\bar{S} = \{s,t\}$. In this case, the Conjectures of Geck, Iancu, Lam and the author [5, Conjectures A et B] suggest the following one:

Conjecture C. *The Conjectures A and B are true for* (W_n,S_n). *The essential hyperplanes are* $\mathscr{H}_s$, $\mathscr{H}_t$, $\mathscr{H}_{t-is}$ $(1 \leqslant i \leqslant n-1)$ *and* $\mathscr{H}_{t+is}$ $(1 \leqslant i \leqslant n-1)$.

COMMENT - The reader may refer to the papers [7], [1], [5], [2] et [6] (in chronological order) for results, comments, evidences for this conjecture. Note also the work of Pietraho [19] in relation with Conjecture B. $\square$

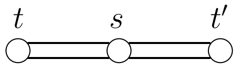
5.E. **Type $\widetilde{G}_2$.** Assume in this subsection, and only in this subsection, that $S = \{t, s_1, s_2\}$, that (W,S) is a Coxeter system of type $\widetilde{G}_2$, and that the Coxeter graph of (W,S) is given by:



Let $s = \{s_1, s_2\}$ and let us identify t and $\{t\}$, so that $\bar{S} = \{s,t\}$. J. Guilhot proved that Conjecture A holds in this case (see [14] and [13]):

Theorem 5.5 (Guilhot). *If (W,S) is an affine Weyl group of type $\widetilde{G}_2$, then the Conjecture A holds. With the above notation, the essential hyperplanes are* $\mathscr{H}_s$, $\mathscr{H}_t$, $\mathscr{H}_{s-t}$, $\mathscr{H}_{s+t}$, $\mathscr{H}_{2t-3s}$, $\mathscr{H}_{2t+3s}$, $\mathscr{H}_{t-2s}$ *and* $\mathscr{H}_{t+2s}$.

5.F. **Type $\widetilde{\boldsymbol{B}}_{\mathbf{2}}$.** Assume in this subsection, and only in this subsection, that $S = \{t, t', s\}$, that (W, S) is a Coxeter system of type $\widetilde{B}_2$, and that the Coxeter graph of (W, S) is given by:



Let us identify t, t' and s with $\{t\}$, $\{t'\}$ and $\{s\}$ respectively, so that $\bar{S} = \{s, t, t'\}$. J. Guilhot proved that Conjecture A holds in this case (see [14] and [13]):

Theorem 5.6 (Guilhot)**.** *If (W, S) is an affine Weyl group of type $\widetilde{B}_2$, then the Conjecture A holds. With the above notation, the essential hyperplanes are $\mathscr{H}_s$, $\mathscr{H}_t$, $\mathscr{H}_{t'}$, $\mathscr{H}_{s-t}$, $\mathscr{H}_{s+t}$, $\mathscr{H}_{s-t'}$, $\mathscr{H}_{s+t'}$, $\mathscr{H}_{t-t'}$, $\mathscr{H}_{t+t'}$, $\mathscr{H}_{s+t+t'}$, $\mathscr{H}_{s+t-t'}$, $\mathscr{H}_{s-t+t'}$, $\mathscr{H}_{s-t-t'}$, $\mathscr{H}_{2s+t+t'}$, $\mathscr{H}_{2s+t-t'}$, $\mathscr{H}_{2s-t+t'}$ and $\mathscr{H}_{2s-t-t'}$.*

This is the only case where $|\bar{S}| = 3$ and where the decomposition into cells have been obtained for all choices of parameters.

5.G. **General results about finite Coxeter groups.** Though it is stated in a different context, the next result is proved by Geck in [11, §3].

Proposition 5.7 (Geck)**.** *If W is finite, then the Conjecture A (a) holds.*

APPENDIX A. POSITIVE SUBSETS OF AN ABELIAN GROUP

In this appendix, we fix a free $\mathbb{Z}$-module Λ of finite rank and we set $V = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$.

The aim of this Appendix is to gather some properties of *positive subsets* of Λ (as defined in [4, §1]). We recall how the set of such positive subsets can be endowed with a topology, and some particular features of this topology.

Definitions, preliminaries. A subset X of Λ is called *positive* if the following three conditions are fulfilled:

(P1) $\Lambda = X \cup (-X)$.
(P2) $X + X \subset X$.
(P3) $X \cap (-X)$ is a subgroup of Λ.

Let $\mathscr{P}\!os(\Lambda)$ denote the set of positive subsets of Λ. If Γ is a totally ordered group and if $\varphi : \Lambda \to \Gamma$ is a morphism of groups, we set

$$\operatorname{Pos}(\varphi) = \{\lambda \in \Lambda \mid \varphi(\lambda) \geqslant 0\}.$$

It is then clear that

(A.1) $$\operatorname{Ker}\varphi = \operatorname{Pos}(\varphi) \cap \operatorname{Pos}(-\varphi) = \operatorname{Pos}(\varphi) \cap - \operatorname{Pos}(\varphi)$$

and that

(A.2) $\operatorname{Pos}(\varphi)$ *is a positive subset of* Λ.

The converse of property A.2 holds. Indeed, let $X \in \mathscr{P}\!os(\Lambda)$ and let $\operatorname{can}_X : \Lambda \to \Lambda/(X \cap (-X))$ denote the canonical morphism; if γ and γ' belong to $\Lambda/(X \cap (-X))$, we shall write $\gamma \leqslant_X \gamma'$ if there exists a representative of $\gamma' - \gamma$ which belongs to X. It is easily seen that

(A.3) $\gamma \leqslant_X \gamma'$ *if and only if all the representatives of* $\gamma' - \gamma$ *belong to* X.

We then deduce immediately from properties (P1), (P2) and (P3) of positive subsets that

(A.4) $(\Lambda/(X \cap (-X)), \leqslant_X)$ *is a totally ordered abelian group*

and that

(A.5) $$X = \operatorname{Pos}(\operatorname{can}_X).$$

Functoriality. If $\sigma : \Lambda \to \Lambda'$ is a morphism between two free abelian groups of finite rank, then [4, Lemma 1.2] the map

$$\begin{array}{rccc} \sigma^* : & \mathscr{P}\!os(\Lambda') & \longrightarrow & \mathscr{P}\!os(\Lambda) \\ & X & \longmapsto & \sigma^{-1}(X) \end{array}$$

is well-defined. This implies that $\mathscr{P}\!os$ is a contravariant functor from the category of free $\mathbb{Z}$-modules of finite rank to the category of sets [4, §1.2].

Linear forms. Let V^* denote the dual of V (i.e. $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$). If $\varphi \in V^*$, we shall denote for simplification $\mathrm{Pos}(\varphi)$ the positive subset $\mathrm{Pos}(\varphi|_\Lambda)$. This defines a map

$$\mathrm{Pos} : V^* \longrightarrow \mathscr{P}os(\Lambda).$$

We recall how to classify the positive subsets of Λ thanks to this application. Let $\mathscr{F}(\Lambda)$ denote the set of finite sequences $(\varphi_1, \dots, \varphi_r)$ such that, for all $i \in \{1, 2, \dots, r\}$, φ_i is a non-zero linear form on $\mathbb{R} \otimes_{\mathbb{Z}} (\Lambda \cap \mathrm{Ker}\, \varphi_{i-1})$ (with the convention that $\varphi_0 = 0$). By convention, we assume that the empty sequence, denoted by $\varnothing$, belongs to $\mathscr{F}(\Lambda)$.

Let $d = \dim V$. Note that, if $(\varphi_1, \dots, \varphi_r) \in \mathscr{F}(\Lambda)$, then $r \leqslant d$. We define the following action of $(\mathbb{R}_{>0})^d$ on $\mathscr{F}(\Lambda)$: if $(\kappa_1, \dots, \kappa_d) \in (\mathbb{R}_{>0})^d$ and if $(\varphi_1, \dots, \varphi_r) \in \mathscr{F}(\Lambda)$, we set

$$(\kappa_1, \dots, \kappa_d) \cdot (\varphi_1, \dots, \varphi_r) = (\kappa_1 \varphi_1, \dots, \kappa_r \varphi_r).$$

Let us endow $\mathbb{R}^r$ with the lexicographic order: then $\mathbb{R}^r$ is a totally ordered abelian group and $(\varphi_1, \dots, \varphi_r) : \Lambda \to \mathbb{R}^r$ is a morphism of groups. So $\mathrm{Pos}(\varphi_1, \dots, \varphi_r)$ is well-defined and belong to $\mathscr{P}os(\Lambda)$ (see A.2). In fact, all the positive subsets of Λ are obtained in this way [4, Proposition 1.10]:

Proposition A.6. *The map*

$$\begin{array}{ccc} \mathscr{F}(\Lambda) & \longrightarrow & \mathscr{P}os(\Lambda) \\ \boldsymbol{\varphi} & \longmapsto & \mathrm{Pos}(\boldsymbol{\varphi}) \end{array}$$

is well-defined and induces a bijection $\mathscr{F}(\Lambda)/(\mathbb{R}_{>0})^d \xrightarrow{\sim} \mathscr{P}os(\Lambda)$.

REMARK - By convention, $\mathrm{Pos}(\varnothing) = \Lambda$. □

Topology on $\mathscr{P}os(\Lambda)$. If E is a subset of Λ, we set

$$\mathscr{U}(E) = \{X \in \mathscr{P}os(\Lambda) \mid X \cap E = \varnothing\}.$$

If $\lambda_1, \dots, \lambda_n$ are elements of Λ, we shall denote for simplification $\mathscr{U}(\lambda_1, \dots, \lambda_n)$ the set $\mathscr{U}(\{\lambda_1, \dots, \lambda_n\})$. If this is necessary, these sets will be denoted by $\mathscr{U}_\Lambda(E)$ or $\mathscr{U}_\Lambda(\lambda_1, \dots, \lambda_n)$. Then

(A.7)
$$\mathscr{U}(E) = \bigcap_{\lambda \in E} \mathscr{U}(\lambda).$$

Note that

(A.8)
$$\mathscr{U}(\varnothing) = \mathscr{P}os(\Lambda) \qquad \text{and} \qquad \mathscr{U}(\Lambda) = \{\varnothing\}.$$

On the other hand, if $(E_i)_{i \in I}$ is a family of subsets of Λ, then

(A.9)
$$\bigcap_{i \in I} \mathscr{U}(E_i) = \mathscr{U}\big(\bigcup_{i \in I} E_i\big).$$

A subset $\mathscr{U}$ of $\mathscr{P}os(\Lambda)$ is called *open* if, for all $X \in \mathscr{U}$, there exists a **finite** subset E of Λ such that $X \in \mathscr{U}(E)$ and $\mathscr{U}(E) \subset \mathscr{U}$ (see [4, §2.1]). The equality A.9 shows that this defines a topology on $\mathscr{P}os(\Lambda)$.

EXAMPLE A.10 - The topological space $\mathscr{P}os(\mathbb{Z})$ has only three points: $\mathbb{Z}$, $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Z}_{\leqslant 0}$. Among them, only $\mathbb{Z}$ is closed whereas $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Z}_{\leqslant 0}$ are open (indeed, $\{\mathbb{Z}_{\geqslant 0}\} = \mathscr{U}(-1)$ and $\{\mathbb{Z}_{\leqslant 0}\} = \mathscr{U}(1)$). □

The following result has been shown in [4, Proposition 2.7]:

Theorem A.11. *The map* $\mathrm{Pos} : V^* \longrightarrow \mathscr{P}os(\Lambda)$ *is continuous and dominant (i.e. its image is dense). Moreover, it induces an homeomorphism between its image and* $V^*/\mathbb{R}_{>0}$.

Recall that, if $\sigma : \Lambda \to \Lambda'$ is a morphism between two free abelian groups of finite rank, then [4, Proposition 2.6] the map $\sigma^* : \mathscr{P}os(\Lambda') \to \mathscr{P}os(\Lambda)$ is continuous. So $\mathscr{P}os$ is a contravariant functor from the category of free abelian groups of finite rank to the category of topological spaces.

Rational subspaces. If E is a subset of Λ, we set

$$\mathscr{L}(E) = \{X \in \mathscr{P}os(\Lambda) \mid E \subset X \cap (-X)\}.$$

Note that

$$\mathrm{Pos}^{-1}(\mathscr{L}(E)) = \{\varphi \in V^* \mid \forall\, \lambda \in E,\ \varphi(\lambda) = 0\} = E^{\perp}.$$

A *rational subspace* of $\mathscr{P}os(\Lambda)$ is a subset of $\mathscr{P}os(\Lambda)$ of the form $\mathscr{L}(E)$, where E is a subset of Λ. If $\lambda \in \Lambda \setminus \{0\}$, we shall denote by $\mathscr{H}_\lambda$ the rational subspace $\mathscr{L}(\{\lambda\})$: such a rational subspace is called a *rational hyperplane.* Note that

$$\mathscr{P}os(\Lambda) = \mathscr{U}(\lambda) \,\dot\cup\, \mathscr{H}_\lambda \,\dot\cup\, \mathscr{U}(-\lambda). \tag{A.12}$$

Half-spaces. Let $\mathscr{H}$ be a rational hyperplane of $\mathscr{P}os(\Lambda)$ and let $\lambda \in \Lambda\setminus\{0\}$ be such that $\mathscr{H} = \mathscr{H}_\lambda$. By A.12, the hyperplane $\mathscr{H}$ defines a unique equivalence relation $\smile_{\mathscr{H}}$ on $\mathscr{P}os(\Lambda)$ such that the equivalence classes are $\mathscr{U}(\lambda)$, $\mathscr{H}$ and $\mathscr{U}(-\lambda)$: note that this relation does not depend on the choice of λ. Moreover [4, Proposition 3.2]:

Proposition A.13. $\mathscr{H}$ *is a closed subset of* $\mathscr{P}os(\Lambda)$ *and* $\mathscr{U}(\lambda)$ *and* $\mathscr{U}(-\lambda)$ *are the connected components of* $\mathscr{P}os(\Lambda) \setminus \mathscr{H}$. *Moreover*

$$\overline{\mathscr{U}(\lambda)} = \mathscr{U}(\lambda) \cup \mathscr{H}_\lambda.$$

Hyperplane arrangements. From now on, and until the end of this Appendix, we shall work under the following hypothesis:

*We fix a **finite** set* $\mathfrak{A}$ *of rational hyperplanes of* $\mathscr{P}os(\Lambda)$.

Following [4, §3.3], we shall recall the notions of *facets*, *chambers* and *faces* associated to $\mathfrak{A}$, in a similar way as these notions are defined for "real" hyperplane arrangements [8, Chapitre V, §1].

Let $\smile_{\mathfrak{A}}$ denote the equivalence relation on $\mathscr{P}os(\Lambda)$ defined as follows: if X and Y are two elements of $\mathscr{P}os(\Lambda)$, we shall write $X \smile_{\mathfrak{A}} Y$ if $X \smile_{\mathscr{H}} Y$ for all $\mathscr{H} \in \mathfrak{A}$. We shall call *facets* (or $\mathfrak{A}$-*facets*) the equivalence classes for the relation $\smile_{\mathfrak{A}}$. We shall call *chambers* (or $\mathfrak{A}$-*chambers*) the facets which meet no hyperplane of $\mathfrak{A}$. If $\mathscr{F}$ is a facet, we set

$$\mathscr{L}_{\mathfrak{A}}(\mathscr{F}) = \bigcap_{\substack{\mathscr{H} \in \mathfrak{A} \\ \mathscr{F} \subset \mathscr{H}}} \mathscr{H},$$

with the usual convention that $\mathscr{L}_{\mathfrak{A}}(\mathscr{F}) = \mathscr{P}os(\Lambda)$ if $\mathscr{F}$ is a chamber. This rational subspace will be called the *support* of $\mathscr{F}$ and we define the *dimension* of $\mathscr{F}$ to be the non-negative integer

$$\dim \mathscr{F} = \dim_{\mathbb{R}} \operatorname{Pos}^{-1}(\mathscr{L}_{\mathfrak{A}}(\mathscr{F})).$$

Similarly, we shall call the *codimension* of $\mathscr{F}$ the non-negative integer

$$\operatorname{codim} \mathscr{F} = \dim_{\mathbb{R}} V - \dim \mathscr{F}.$$

With these definitions, a chamber is a facet of codimension 0. The next proposition has been proved in [4, Proposition 3.3]. The set of $\mathfrak{A}$-facets (respectively $\mathfrak{A}$-chambers) is denoted by $\operatorname{Fac}(\mathfrak{A})$ (respectively $\operatorname{Cham}(\mathfrak{A})$).

Proposition A.14. *Let $\mathscr{F}$ be a facet and let $X \in \mathscr{F}$. Then:*

(a) $\mathscr{F} = \bigcap_{\mathscr{H} \in \mathfrak{A}} \mathscr{D}_{\mathscr{H}}(X)$.

(b) $\overline{\mathscr{F}} = \bigcap_{\mathscr{H} \in \mathfrak{A}} \overline{\mathscr{D}_{\mathscr{H}}(X)}$.

(c) *$\overline{\mathscr{F}}$ is the union of $\mathscr{F}$ and of facets of strictly smaller dimension.*

(d) *If $\mathscr{F}'$ is a facet such that $\overline{\mathscr{F}} = \overline{\mathscr{F}}'$, then $\mathscr{F} = \mathscr{F}'$.*

We define a relation $\preccurlyeq$ on the set of facets: we write $\mathscr{F} \preccurlyeq \mathscr{F}'$ if $\overline{\mathscr{F}} \subseteq \overline{\mathscr{F}}'$ (i.e. if $\mathscr{F} \subseteq \overline{\mathscr{F}}'$). The Proposition A.14 (d) shows that:

Corollary A.15. *The relation $\preccurlyeq$ is a partial order.*

Recall that a map $\xi : \mathscr{X} \to P$, where $\mathscr{X}$ is a topological space and P is a partially ordered set, is called *upper semicontinuous* if, for all $p \in P$, the set $\{x \in \mathscr{X} \mid \xi(x) < p\}$ is open.

Corollary A.16. *Let P be a partially ordered set and let $\xi : \mathscr{P}os(\Lambda) \to P$ be a map satisfying the following properties:*

(1) *ξ is constant on facets (if $\mathscr{F}$ is a facet, we denote by $\xi(\mathscr{F})$ the value of ξ on $\mathscr{F}$).*
(2) *If $\mathscr{F}$ and $\mathscr{F}'$ are two facets such that $\mathscr{F} \preccurlyeq \mathscr{F}'$, then $\xi(\mathscr{F}) \geqslant \xi(\mathscr{F}')$.*

Then ξ is upper semicontinuous.

Proof. Let $p \in P$ and let $\mathscr{U} = \{x \in \mathscr{P}os(\Lambda) \mid \xi(x) < p\}$. Let $x \in \mathscr{U}$ and let $\mathfrak{A}_x = \{\mathscr{H} \in \mathfrak{A} \mid x \notin \mathscr{H}\}$. Let $\mathscr{C}$ be the $\mathfrak{A}_x$-facet containing x. Then $\mathscr{C}$ is an $\mathfrak{A}_x$-chamber (because x does not belong to any hyperplane in $\mathfrak{A}_x$), and so $\mathscr{C}$ is open. To prove the corollary, it is enough to show that $\mathscr{C} \subseteq \mathscr{U}$.

For this, let $y \in \mathscr{C}$ and let $\mathscr{F}'$ be the $\mathfrak{A}$-facet containing y. Then $\mathscr{F}' \subseteq \mathscr{C}$ and so it is sufficient to show that $\mathscr{F}' \subseteq \mathscr{U}$. Let $\mathscr{F}$ denote the $\mathfrak{A}$-facet of x: by the properties (1) and (2), we only need to show that $\mathscr{F} \preccurlyeq \mathscr{F}'$.

Then, by Corollary A.14,

$$\overline{\mathscr{F}'} = \Bigl(\bigcap_{\mathscr{H} \in \mathfrak{A}_x} \overline{\mathscr{D}_{\mathscr{H}}(y)}\Bigr) \bigcap \Bigl(\bigcap_{\mathscr{H} \in \mathfrak{A} \setminus \mathfrak{A}_x} \overline{\mathscr{D}_{\mathscr{H}}(y)}\Bigr)$$

and
$$\overline{\mathscr{F}} = \Bigl(\bigcap_{\mathscr{H} \in \mathfrak{A}_x} \mathscr{D}_{\mathscr{H}}(x)\Bigr) \bigcap \Bigl(\bigcap_{\mathscr{H} \in \mathfrak{A} \setminus \mathfrak{A}_x} \overline{\mathscr{D}_{\mathscr{H}}(x)}\Bigr).$$

But, since x and y belongs to $\mathscr{C}$, we have, again by Corollary A.14 (but applied to $\mathfrak{A}_x$),

$$\bigcap_{\mathscr{H} \in \mathfrak{A} \setminus \mathfrak{A}_x} \overline{\mathscr{D}_{\mathscr{H}}(y)} = \bigcap_{\mathscr{H} \in \mathfrak{A} \setminus \mathfrak{A}_x} \overline{\mathscr{D}_{\mathscr{H}}(x)} = \overline{\mathscr{C}}.$$

So
$$\overline{\mathscr{F}'} = \overline{\mathscr{C}} \bigcap \Bigl(\bigcap_{\mathscr{H} \in \mathfrak{A} \setminus \mathfrak{A}_x} \overline{\mathscr{D}_{\mathscr{H}}(y)}\Bigr)$$

and
$$\overline{\mathscr{F}} = \overline{\mathscr{C}} \bigcap \Bigl(\bigcap_{\mathscr{H} \in \mathfrak{A} \setminus \mathfrak{A}_x} \mathscr{H}\Bigr).$$

Now, if $\mathscr{H} \in \mathfrak{A}_x$, then $\overline{\mathscr{D}_{\mathscr{H}}(y)}$ contains $\mathscr{H}$, so $\overline{\mathscr{F}} \subseteq \overline{\mathscr{F}'}$, as expected. $\square$

REMARK - Under the hypothesis of the Proposition, the above proof can be followed word by word to show that the set $\{x \in \mathscr{P}os(\Lambda) \mid \xi(x) \leqslant p\}$ is also open. $\square$

REFERENCES

[1] C. BONNAFÉ, Two-sided cells in type B (asymptotic case), *J. Algebra* **304** (2006), 216-236.

[2] C. BONNAFÉ, On Kazhdan-Lusztig cells in type B, *Journal of Alg. Combinatorics*, **31** (2010), 53-82.

[3] C. BONNAFÉ & M. DYER, Semidirect product decomposition of Coxeter groups, preprint (2008), to appear in *Comm. in Algebra*, available at `arXiv:0805.4100`.

[4] C. BONNAFÉ, Topologie sur l'ensemble des parties positives d'un réseau, *Ann. Sc. Math. Québec* **32** (2008), 11-27.

[5] C. BONNAFÉ, M. GECK, L. IANCU & T. LAM, On domino insertion and Kazhdan-Lusztig cells in type B_n, to appear in *Progress in Mathematics* (Birkhäuser), available at `arXiv:math/0609279`.

[6] C. BONNAFÉ & J. GUILHOT, On Kazhdan-Lusztig cells in the asymptotic case, *in preparation.*

[7] C. BONNAFÉ & L. IANCU, Left cells in type B_n with unequal parameters, *Represent. Theory* **7** (2003), 587–609.

[8] N. BOURBAKI, *Groupes et algèbres de Lie, chapitres IV, V et VI*, Hermann, Paris, 1968.

[9] M. CHLOUVERAKI, *Blocks and families for Cyclotomic Hecke algebras*, Lecture Notes in Mathematics **1981**, 2009, Springer.

[10] M. GECK, On the induction of Kazhdan-Lusztig cells, *Bull. London Math. Soc.* **35** (2003), 608-614.

[11] M. GECK, Computing Kazhdan-Lusztig cells for unequal parameters, *J. Algebra* **281** (2004), 342–365.

[12] M. GECK, G. HISS, F. LÜBECK, G. MALLE & G. PFEIFFER, CHEVIE - A system for computing and processing generic character tables, *Appl. Algebra Engrg. Comm. Comput.* **7** (1996) 175-210; available at `http://www.math.rwth-aachen.de/~CHEVIE`.

[13] J. GUILHOT, Computations of Kazhdan-Lusztig cells in type $\widetilde{G}_2$: the unequal parameter case, Ph.D. Thesis (2008), Aberdeen and Lyon.

[14] J. GUILHOT, Generalized induction of Kazhdan-Lusztig cells, *Ann. Inst. Four.* **59** (2009), 1349–1376.

[15] J. GUILHOT, Kazhdan-Lusztig cells in the affine Weyl groups of rank 2, preprint (2009), available at `arXiv:0901.1711`.

[16] D. A. KAZHDAN AND G. LUSZTIG, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[17] S. GAL, On normal subgroups of Coxeter groups generated by standard parabolic subgroups, *Geom. Ded.* **115** (2005), 65–78.

[18] G. LUSZTIG, *Hecke algebras with unequal parameters*, CRM Monograph Series **18**, American Mathematical Society, Providence, RI (2003), 136 pp.

[19] T. PIETRAHO, Cells and constructible representations in type B, *New York J. Math.* **14** (2008), 411–430.

LABO. DE MATH. DE BESANÇON (CNRS: UMR 6623), UNIVERSITÉ DE FRANCHE-COMTÉ, 16 ROUTE DE GRAY, 25030 BESANÇON CEDEX, FRANCE

E-mail address: `cedric.bonnafe@univ-fcomte.fr`

URL: `http://www-math.univ-fcomte.fr/pp_Annu/CBONNAFE/`